Distance Optimization

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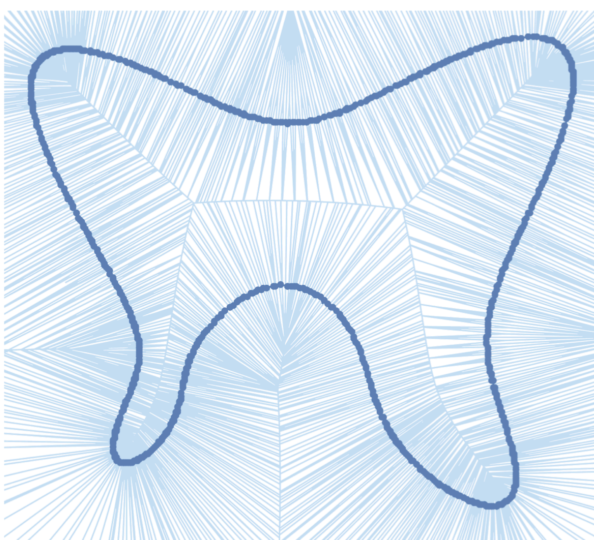
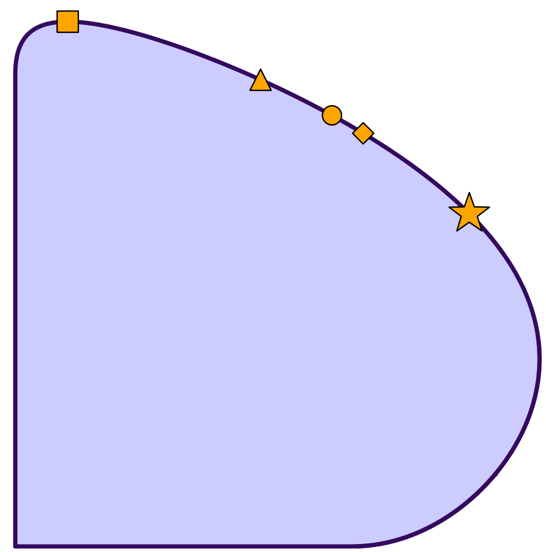
April 15, Metric Algebraic Geometry: Starting Local





Two Distance Optimization Perspectives

Let $\mathscr{M} \subseteq \mathbb{C}^n$ be an algebraic variety.

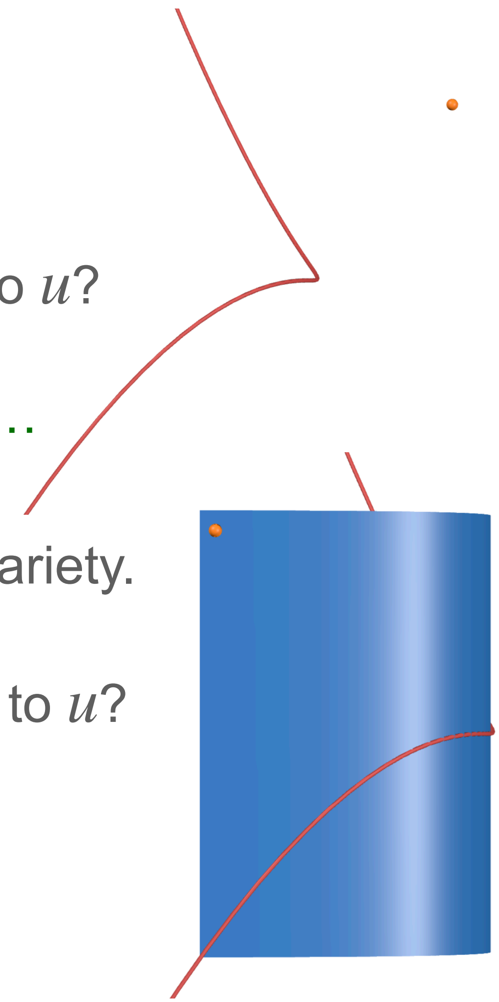
Given a data point $u \in \mathbb{R}^n$, what point on $\mathscr{M}_{\mathbb{R}}$ is closest to u?

Euclidean distance, Kullback-Leibler divergence, …

Let $\mathscr{V}$ be an algebraic variety, and let $\mathscr{M} \subseteq \mathscr{V}$ be a subvariety.

Given a data point $u \in \mathscr{V}_{\mathbb{R}}$, what point on $\mathscr{M}_{\mathbb{R}}$ is closest to u?

Euclidean distance, geodesics, …



The Euclidean Distance Degree

Let $\mathscr{M} = V(f_1, \ldots, f_r) \subseteq \mathbb{C}^n$ be an algebraic variety of codimension c.

Given a data point $u \in \mathbb{R}^n$, what point on $\mathscr{M}$ is closest to u?

$$\min_{x \in \mathscr{M}_{\mathbb{R}}} ||x - u||^2 = \sum_{i=1}^{n} (x_i - u_i)^2$$

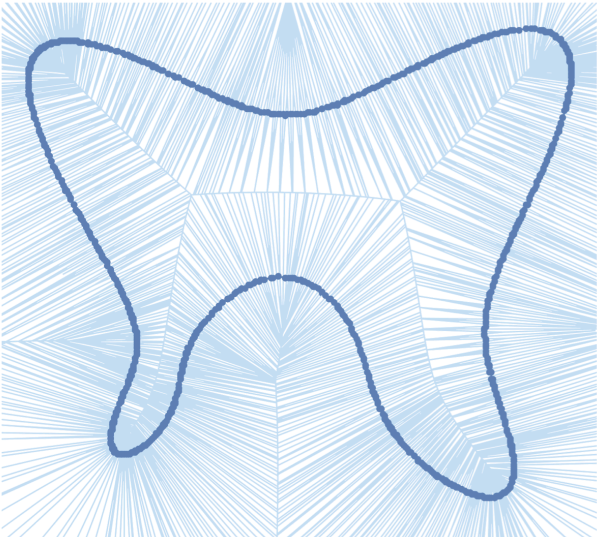
Critical ideal:

$$(\langle f_1, \ldots, f_r \rangle + \langle (c+1) \times (c+1) \text{ minors of } \mathscr{J}^u \rangle) : I^{\infty}_{\mathrm{Sing}(\mathscr{M})}$$

$$\mathscr{J}^u = \begin{pmatrix} (x-u)^T \\ -\nabla f_1(x)- \\ \vdots \\ -\nabla f_r(x)- \end{pmatrix}$$

For generic data, this ideal is zero dimensional.
The ED degree of $\mathscr{M}$ is its degree.

Part I: Voronoi Cells



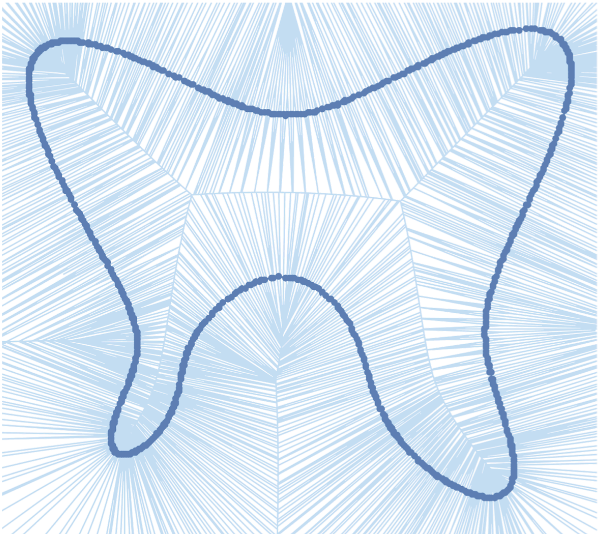
Voronoi Cells

Let $\mathscr{M} \subseteq \mathscr{V} \subseteq \mathbb{C}^n$ be an algebraic variety.

Given a data point $u \in \mathbb{C}^n$, what point on $\mathscr{M}$ is closest to u?

Given a model point $p \in \mathscr{M}$, for what points on $u \in \mathbb{C}^n$ is p the answer to the above question?

Given a model point $p \in \mathscr{M}$, for what points on $u \in \mathscr{V}$ is p the answer to the above question?



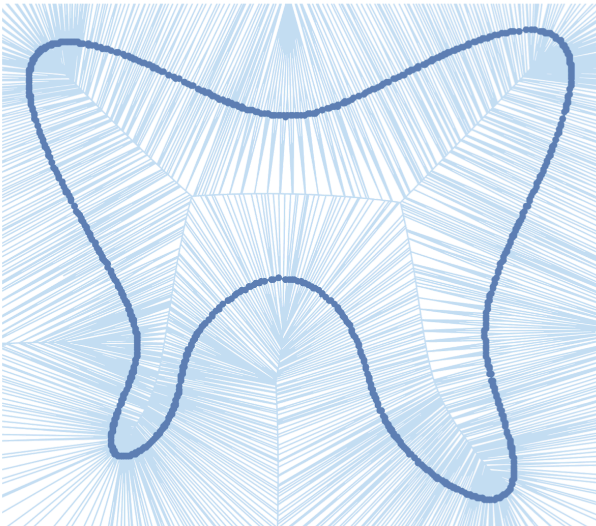
Voronoi Cells

The Voronoi cells of a point $p \in \mathscr{M}$ are

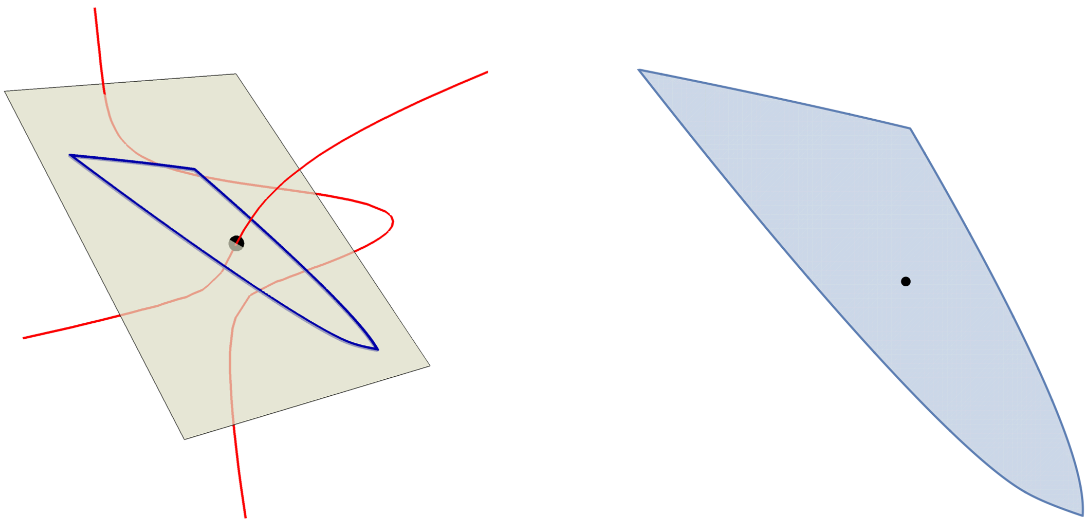
$$\{u \in \mathbb{R}^n : p \in \text{argmin}_{p \in \mathscr{M}} ||u - p||^2\}$$

$$\{u \in \mathscr{V} : p \in \text{argmin}_{p \in \mathscr{M}} ||u - p||^2\}$$

The points u that are equidistant from two points of $\mathscr{M}$ form the *medial axis* of $\mathscr{M}$.



The Voronoi Boundary of a Point



Computing the Voronoi Boundary

Recall: x is critical for u if it is in the vanishing set of this ideal:

$$I_u(x) = (\langle f_1, \ldots, f_r \rangle + \langle (c+1) \times (c+1) \text{ minors of } \mathcal{J}^u \rangle) : I^{\infty}_{\mathrm{Sing}(\mathcal{M})} \qquad \mathcal{J}^u = \begin{pmatrix} (x-u)^T \\ -\nabla f_1(x)- \\ \vdots \\ -\nabla f_r(x)- \end{pmatrix}$$

The Voronoi boundary at the point p is the zero set of the elimination ideal

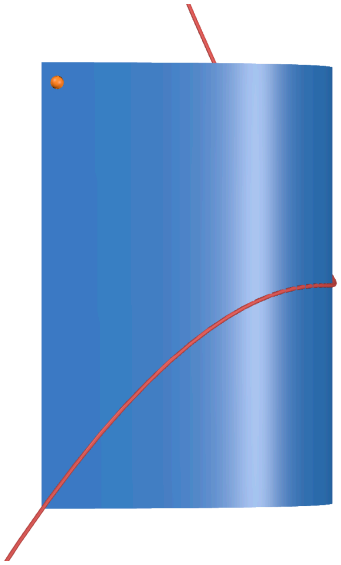
$$((I_u(x) \; + I_u(p) \; + \langle\, ||x-u||^2 - ||p-u||^2 \rangle) : \langle x - p \rangle^{\infty}) \cap \mathbb{C}[u_1, \ldots, u_n] \,.$$

Or one can sample the Voronoi cell by generating data u such that $p - u$ is the rowspan of $\begin{pmatrix} -\nabla f_1(p)- \\ \vdots \\ -\nabla f_r(p)- \end{pmatrix}$ and then checking if p is the closest point to u.

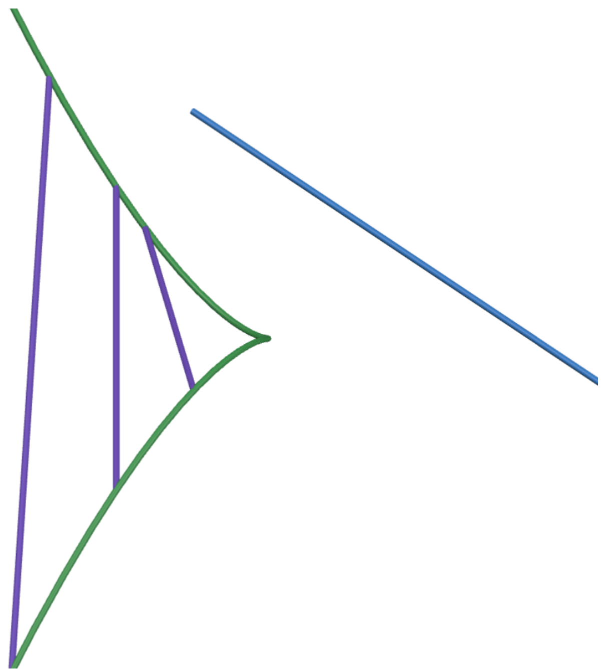
The Degree of the Voronoi Boundary

Theorem (Cifuentes-Ranestad-Sturmfels-Weinstein, 2022).
Let $X \subseteq \mathbb{P}^n$ be a curve in general position of degree d and geometric genus g with only mild singularities. The degree of boundary of the Voronoi cell at a general point is $4d + 2g - 6$.

Part II: The Grassmann Distance Degree



Lines in 3-Space



Which secant line is closest to the data line?

The (Projection) Grassmannian

Today we represent points in the Grassmannian by symmetric projection matrices.

$$\mathrm{pGr}_{\mathbb{R}}(k,n) = \{P \in \mathrm{Sym}^2\mathbb{R}^n : P^2 = P, \mathrm{trace}(P) = k\}$$

$$\mathrm{pGr}(2,n) = \{P \in \mathrm{Sym}^2\mathbb{C}^n : P^2 = P, \mathrm{trace}(P) = 2\}$$

This is a smooth, irreducible affine variety of dimension $2(n-2)$.

It is the $\mathrm{O}(\mathbb{C}^n)$ orbit of the matrix $\begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & 0 \end{pmatrix}$.

Euclidean Distance Degree

Let $\mathscr{M} \subseteq \mathrm{pGr}(2,n)$ be a subvariety of the projection Grassmannian.

The Euclidean distance degree of $\mathscr{M}$ is the number of critical points of the optimization problem

$$\min_{P \in \mathscr{M}} ||P - Q||_F^2 \quad \Longleftrightarrow \quad \max_{P \in \mathscr{M}} \mathrm{trace}(PQ)$$

for a general symmetric matrix $Q \in \mathrm{Sym}^2\mathbb{R}^n$.

Grassmann Distance Degree

Let $\mathscr{M} \subseteq \mathrm{pGr}(2,n)$ be a subvariety of the projection Grassmannian.

The Grassmann distance degree of $\mathscr{M}$ is the number of critical points of the optimization problem

$$\min_{P \in \mathscr{M}} ||P - Q||_F^2 \quad \Longleftrightarrow \quad \max_{P \in \mathscr{M}} \mathrm{trace}(PQ)$$

for a general projection matrix $Q \in \mathrm{pGr}(2,n)$.

Extraneous Critical Points

Extraneous critical points are critical for data in the Grassmannian no matter what the model is.

$$\max_{P \in \mathscr{M}} \operatorname{trace}(PQ) \qquad\qquad Q \in \operatorname{pGr}(2, n)$$

Proposition (F-Rosana-Sturmfels, 2026). The Q-invariant subspaces of dimension 2 are always critical points, regardless of what $\mathscr{M}$ is. Each of these critical points P satisfies $PQ = QP$.

Theorem (F-Rosana-Sturmfels, 2026). For $n \geq 6$ and a model $\mathscr{M}$ of dimension at least 4, the ED degree of $\mathscr{M}$ is strictly larger than the GD degree of $\mathscr{M}$.

Measuring Distance

Theorem (Ye-Lim, 2016). Any nice (rotation-invariant) distance between two subspaces in $\mathrm{Gr}(2,n)$ depends only on the *principal angles* between the subspaces.

Principal Angles:

If P and Q are symmetric projection matrices onto 2-dimensional subspaces of $\mathbb{R}^n$, then the matrix PQ has 2 positive eigenvalues $\lambda \geq \mu$.

The principal angles are $\theta_1 = \cos^{-1}(\sqrt{\lambda})$ and $\theta_2 = \cos^{-1}(\sqrt{\mu})$.

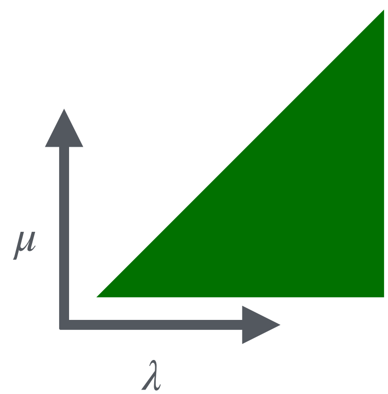
Upshot: the goal of any distance minimization problem is to simultaneously minimize the principal angles.

Measuring Distance

Given a model $\mathscr{M} \subseteq \mathrm{Gr}(2,n)$ and a data point $Q \in \mathrm{pGr}(2,n)$, there is a function

$$\mathscr{M}_{\mathbb{R}} \to \mathbb{R}^2 \qquad P \mapsto (\lambda, \mu)$$

where $1 \geq \lambda \geq \mu > 0$ are the nonzero eigenvalues of PQ. The image of this map is the *spectral region*. This is a 2-dimensional subset of the semi algebraic set



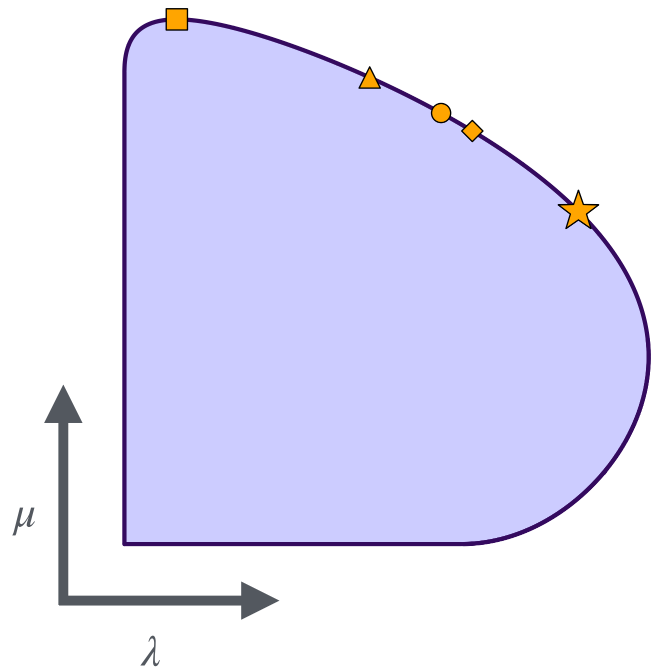
Example

$$P = \frac{1}{x^2+y^2+z^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & x^2 & xy & xz & 0 \\ 0 & xy & y^2 & yz & 0 \\ 0 & xz & yz & z^2 & 0 \\ 0 & 0 & 0 & 0 & x^2+y^2+z^2 \end{pmatrix}$$

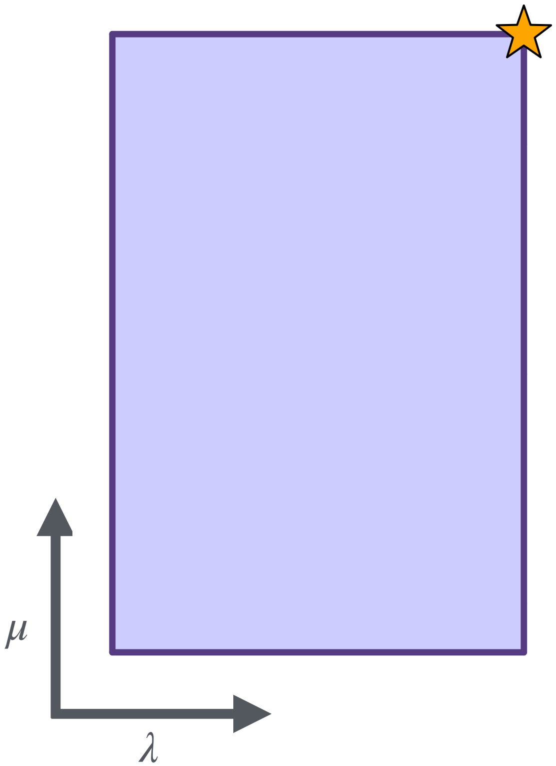
$$Q = \frac{1}{10}\begin{pmatrix} 6 & 4 & 2 & 0 & -2 \\ 4 & 3 & 2 & 1 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ -2 & 0 & 2 & 3 & 6 \end{pmatrix}$$

The Spectral Region

	Distance squared in λ, μ	Minimizer
Chordal	$4 - 2\lambda - 2\mu$	☆
Geodesic	$\arccos(\sqrt{\lambda})^2 + \arccos(\sqrt{\mu})^2$	○
Procrustes	$4 - 2\sqrt{\lambda} - 2\sqrt{\mu}$	◇
Binet-Cauchy	$1 - \lambda\mu$	△
Fubini-Study	$\arccos(\sqrt{\lambda\mu})^2$	△
Martin	$-\log(\lambda\mu)$	△
Asimov	$\arccos(\sqrt{\mu})^2$	□
Projection	$1 - \mu$	□
Spectral	$2 - 2\sqrt{\mu}$	□



Snugness



All distances have the same maximal eigenvalues!

Models with this property are called ***snug***.

Theorem (F-Rosana-Sturmfels, 2026).

The following Schubert varieties are snug:

$$\begin{pmatrix} 1 & * & \cdots & 0 & \cdots & * \\ 0 & 0 & \cdots & 1 & \cdots & * \end{pmatrix} \qquad \begin{pmatrix} 0 & \cdots & 1 & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 1 & * & \cdots & * \end{pmatrix}$$

Questions

- Is the spectral region always convex?
- For a given model and data point, what is the degree of the spectral curve? Is it independent of the data point?
- Are there examples of snug models that are not Schubert varieties?
- What can we say about Voronoi cells of the projection Grassmannian?
- If one intersects a Voronoi cell of the projection Grassmannian with the projection Grassmannian, is the resulting set connected? Convex?
- Is there an analog of the spectral region for other varieties?

References

- *Metric Algebraic Geometry* (P. Breiding, K. Kohn, B. Sturmfels)
- *The Euclidean Distance Degree* (J. Draisma, E. Horobeț, G. Ottaviani, B. Sturmfels, R. Thomas)
- *Voronoi Cells of Varieties* (D. Cifuentes, K. Ranestad, B. Sturmfels, M. Weinstein)
- *Voronoi Cells in Metric Algebraic Geometry of Plane Curves* (M. Brandt, M. Weinstein)
- *Distance Optimization in the Grassmannian of Lines* (H. Friedman, A. Rosana, B. Sturmfels)