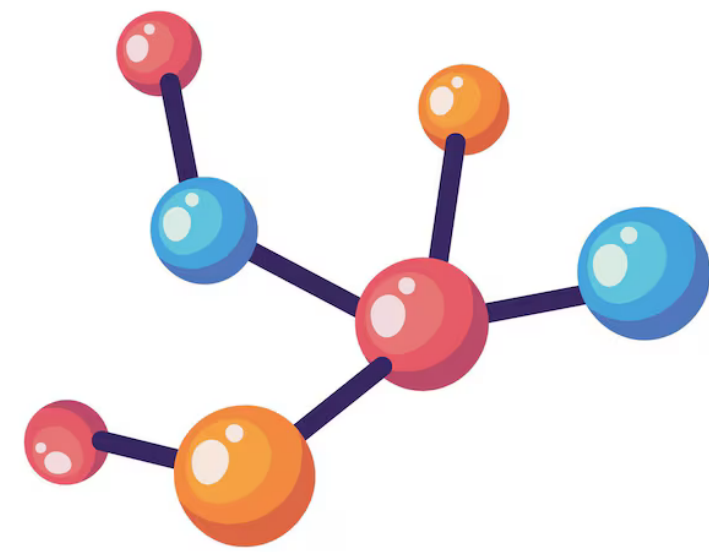


Numerical Algebraic Geometry for Energy Computations on Tensor Train Varieties

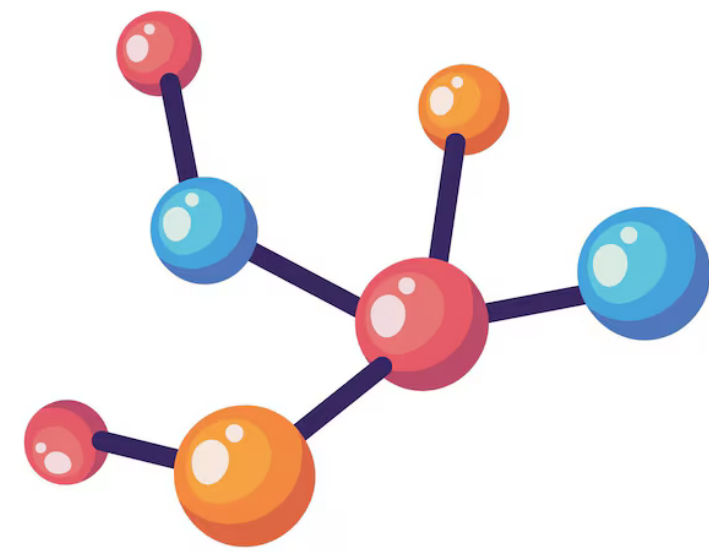
Viktoriia Borovik, Hannah Friedman, Serkan Hoşten, and Max Pfeffer*
Computational and Effective Commutative Algebra, MEGA 2026



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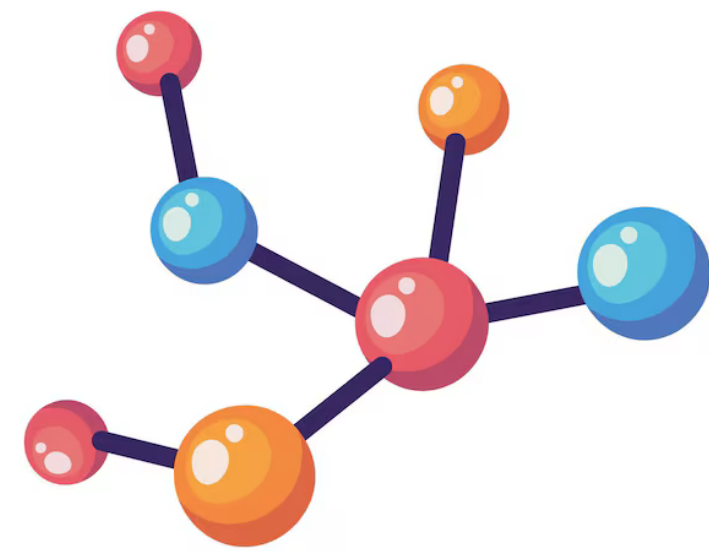


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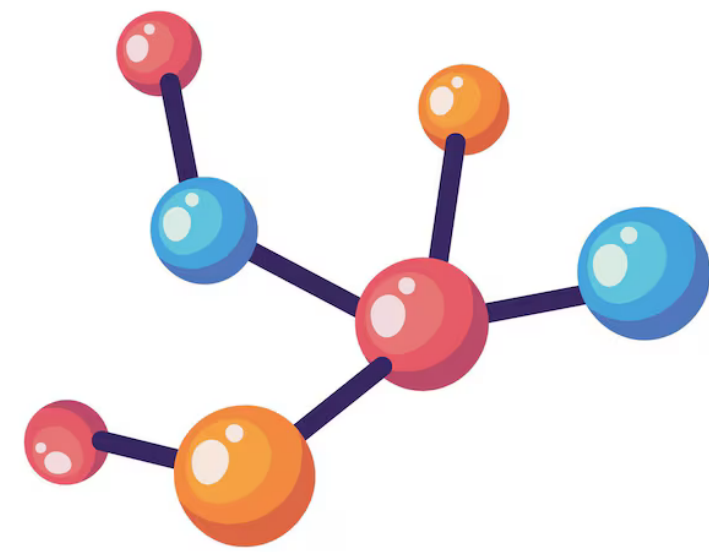
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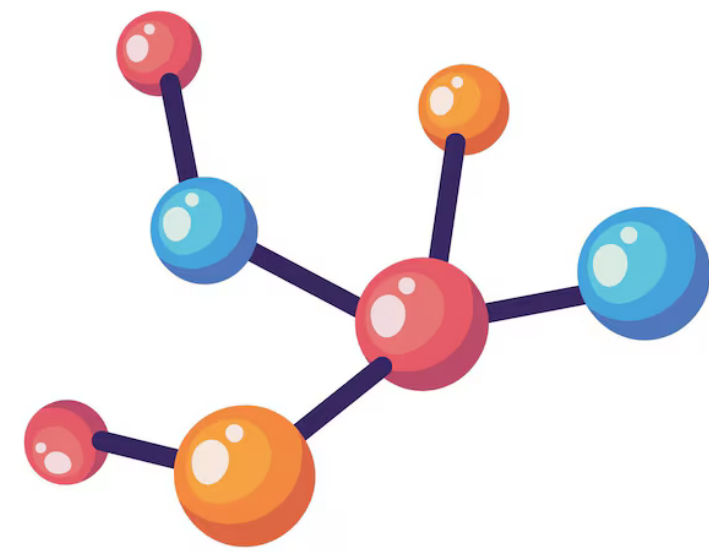


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minimize $\frac{\psi^\dagger H \psi}{\psi^\dagger \psi}$ subject to $\psi \in \mathcal{X} \subseteq \mathbb{P}^{n-1}$.

or

minimize $\frac{\psi^T H \psi}{\psi^T \psi}$ subject to $\psi \in \mathcal{X} \subseteq \mathbb{P}^{n-1}$.

H is a symmetric
 $n \times n$ matrix

Numerical Algebraic Geometry for Energy Computations on Tensor Train Varieties

- Introduction
- Rayleigh-Ritz Optimization and the RR Degree
- Rayleigh-Ritz Correspondence and the Discriminant
- Bombieri-Weyl Correspondence
- Geometry of Tensor Train Varieties
- One-to-one Parametrization of TT Manifolds
- Numerical Experiments

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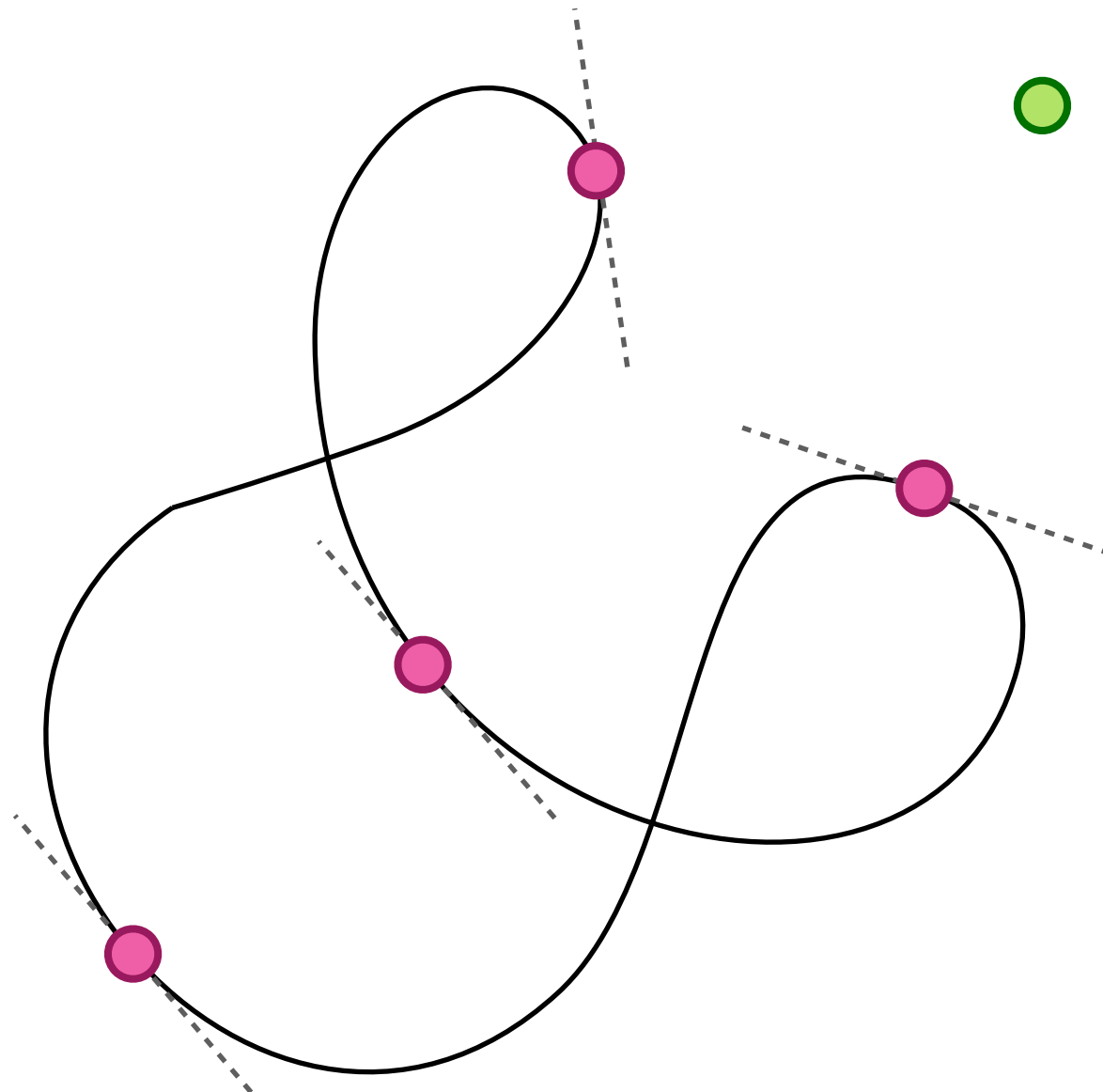
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Nonlinear Kalman Varieties, Salizzoni-Sodomaco*-Weigert
→ Wednesday 11:30

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The RR discriminant is also related to the difficulty of a problem, and governs the number of real critical points of a problem.

Proposition (Borovik-F.-Hoşten-Pfeffer, 2025). The RR degree of $\mathcal{X} = \mathbb{P}^{n-1}$ is n . The critical points are the eigenvectors of H .

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Proposition (Borovik-F.-Hoşten-Pfeffer, 2025). The RR degree of the rational normal curve of degree d , i.e., the image of the map,

$$\mathbb{P}^1 \rightarrow \mathbb{P}^d, \quad [t : s] \mapsto [t^d : t^{d-1}s : \cdots : s^d]$$

is $2(2d - 1)$.

The RR discriminant is a hypersurface of degree $2(4d - 3)$.

Example (Rational normal quadric).

$$\mathbb{P}^1 \rightarrow \mathbb{P}^2$$
$$[t : s] \mapsto [t^2 : ts : s^2]$$

The RR degree is 6.

The RR discriminant is a hypersurface of degree 10.

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The defining polynomial of the RR discriminant has three irreducible factors:

$$3(2h_{12} - 2h_{13} - h_{22} + h_{33})^2 + (2h_{11} + 2h_{12} - 2h_{13} - h_{22} - 4h_{23} - h_{33})^2$$

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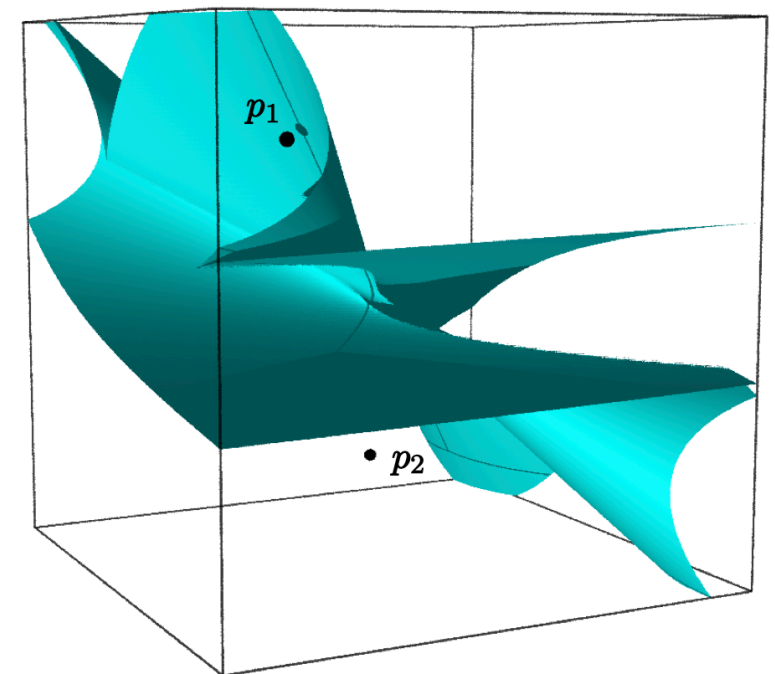
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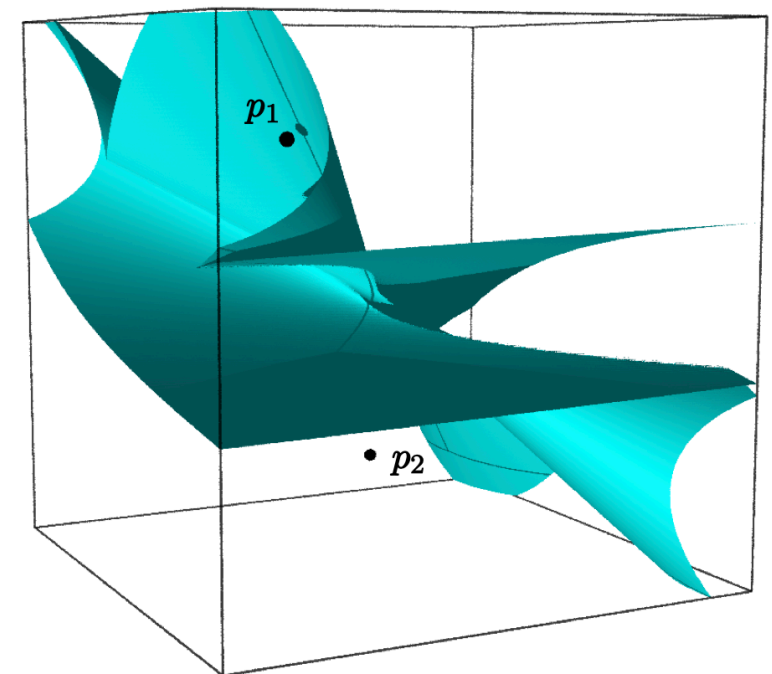
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The number of real critical points is 2 or 4.



(Salizzoni-Sodomaco-Weigert, 2025)

The *RR correspondence* is the incidence variety

$$\mathcal{R}_{\mathcal{X}} = \overline{\{(\psi, H) \in \mathcal{X}_{\text{reg}} \times \text{Sym}^2(\mathbb{C}^n) : \psi \text{ is critical for } H, \psi^T \psi \neq 0\}}$$

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If $I_{\mathcal{R}_{\mathcal{X}}} = \langle g_1, \dots, g_s \rangle \subseteq \mathbb{C}[\psi, H]$, we write $\mathcal{R}_{\mathcal{X}}^{\text{ram}}$ for the pairs (ψ, H) in $\mathcal{R}_{\mathcal{X}}$ where

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The *isotropic* part of the discriminant is $\pi_2(\mathcal{R}_{\mathcal{X}}^{\text{ram}} \cap V(\psi^T \psi))$.

The *nonisotropic* part of the discriminant is $\pi_2(\overline{\mathcal{R}_{\mathcal{X}}^{\text{ram}} \setminus V(\psi^T \psi)})$.

Example ($\mathcal{X} = \mathbb{P}^1$).

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix}$$

The isotropic part of the RR discriminant is equal to the whole discriminant, namely $V((h_{11} - h_{22})^2 + 4h_{12}^2)$.

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Example (Rational normal quadric).

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The RR degree is 6.

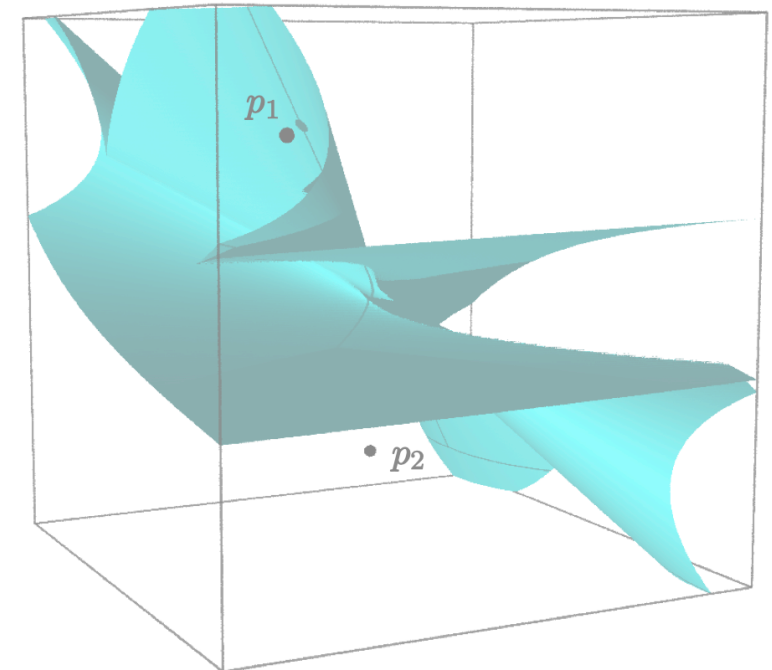
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nonisotropic part

(Salizzoni-Sodomaco-Weigert, 2025)

Conjecture (Borovik-F.-Hoşten-Pfeffer, 2025). The isotropic part of the RR discriminant is a hypersurface. Each irreducible component is cut out by a sum of squares.

Given $\mathbf{k} = (k_0, \dots, k_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$, the tensor train variety $\mathcal{T}_{\mathbf{k}, \mathbf{r}}$ is the set of tensors of format $k_0 \times k_1 \times \dots \times k_n$ such that the flattening matrix with format $k_0 \cdots k_{i-1} \times k_i \cdots k_n$ has rank at most r_i .

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Theorem (Borovik-F.-Hoşten-Pfeffer, 2025). Algorithm 2 defines a one-to-one parametrization

$$\mathrm{Gr}(r_1, k_0) \times \mathrm{Gr}(r_2, r_1 k_1) \times \dots \times \mathrm{Gr}(r_n, r_{n-1} k_{n-1}) \times \mathbb{P}(\mathbb{C}^{r_n \times k_n}) \dashrightarrow \mathcal{T}_{\mathbf{k},\mathbf{r}}$$

of a tensor train variety.

Algorithm 2: Decompression $\mathrm{decompress}(X_0, \dots, X_n)$

Input : $(X_0, \dots, X_n) \in \mathcal{W} \subseteq \mathcal{Z}$

Output: $\psi \in \mathcal{U} \subseteq V_{\mathbf{k},\mathbf{r}}^=$

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1  $\mathbf{r} := (1, r_1, \dots, r_n, 1), \psi := X_n;$ 
2 for  $i = n - 1$  to 0 do
3    $R_i = \mathrm{unfold}(\psi, [r_{i+1}, k_{i+1} \cdots k_n]);$ 
4    $\psi^i = X_i R_i;$ 
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Corollary.

$$\dim(\mathcal{T}_{\mathbf{k},\mathbf{r}}) = r_n k_n - 1 + \sum_{i=0}^{n-1} r_{i+1} (r_i k_i - r_{i+1})$$

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Degree?

(Rosana-Schmidt, 2026)

r	$m \backslash n$	2	3	4	5	6	7	8
1	2	8	18	32	50	72	98	128
1	3		61	148	295	518	833	1256
1	4			480	1220	2624	5012	8768
1	5				3881	10166	23051	46856
2	2	4	6	8	10	12	14	16
2	3		154	448	970	1784	2954	4544
2	4			5840	24924	74775	182306	386716
3	3		9	12	15	18	21	24
3	4			2368	7340	17552	35755	65280
3	5				460351			

Table 1: RR degrees of determinantal varieties of $m \times n$ matrices with rank $\leq r$.

r	$m \setminus n$	2	3	4	5	6	7	8
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k	r	codim	RR deg	time	Disc. deg	time
$(2)_3$	$(1)_2$	5	48	14s	360	1.8m
$(2)_4$	$(1)_3$	12	384	90s	5438	3h
$(2)_4$	$(1, 2, 1)$	10	352	59s	6000	6.4h
$(2)_5$	$(1)_4$	27	3840	26m		
$(2)_5$	$(1, 2, 1, 1)$	25	4608	33m		
$(2)_5$	$(1, 2, 2, 1)$	22	2752	1h		
$(2)_6$	$(1)_5$	58	46080	24h		
$(2)_6$	$(1, 2, (1)_3)$	56	69120	49h		
$(2)_6$	$(1, 1, 2, 1, 1)$	56	69120	37h		
$(2)_6$	$(1, 2, 2, 1, 1)$	53	67583	153h		
$(2)_6$	$(1, 2, 1, 2, 1)$	55	134656	78h		

Table 2: RR degrees and RR discriminant degrees for small tensor train varieties.

r	$m \setminus n$	2	3	4	5	6	7	8
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Table 2: RR degrees and RR discriminant degrees for small tensor train varieties.

Proposition (Borovik-F.-Hoşten-Pfeffer, 2025). The smallest tensor train variety which is not a Segre product is $\mathbf{k} = (2, 2, 2, 2, 2, 2)$ and $\mathbf{r} = (1, 2, 2, 2, 1)$. Its RR degree is $\geq 691\,127$.

M	Extremal eigenvalues	V_1	C_1	V_2	C_2	V_3	C_3	V_4	C_4
1	-4.7543, 2.9994	-3.5692	336	-3.2364	332	-2.4927	204	-1.9931	128
2	-4.6467, 2.5529	-3.0231	568	-3.0762	432				
3	-5.4243, 3.3603	-3.8648	616	-2.8834	384				
4	-5.6478, 3.0117	-4.2191	710	-2.2213	146	-2.4659	144		
5	-4.7984, 3.9884	-4.1527	628	-3.2443	372				

Table 5: ALS statistics for $\mathbf{k} = (2, 2, 2, 2)$ and $\mathbf{r} = (1, 2, 1)$ for symmetric matrices M_1 - M_5 .



Thank you!

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