The Maximum Likelihood Degree of the Grassmannian

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June 25, 2026

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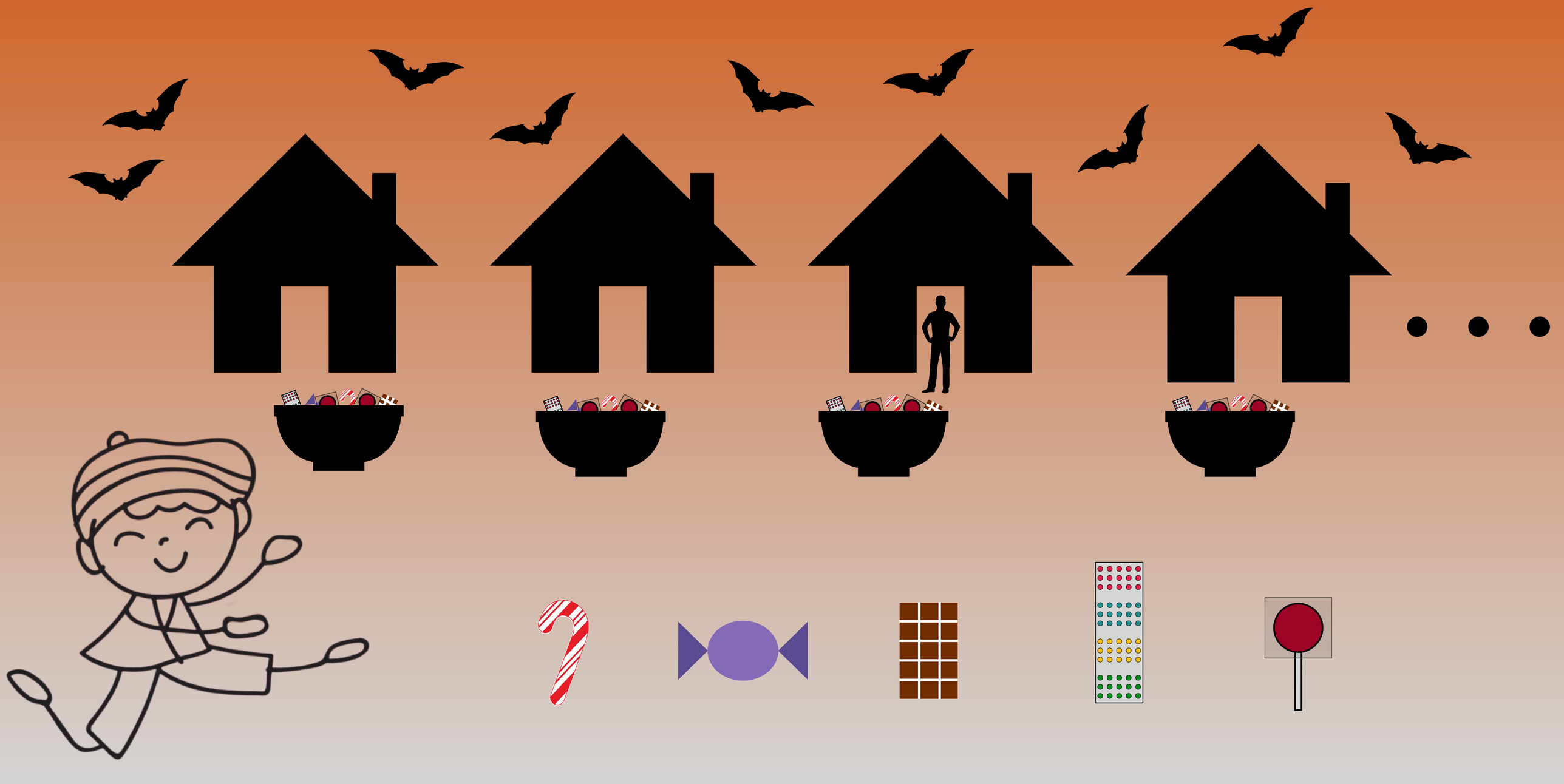
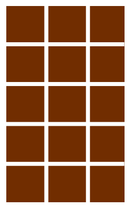
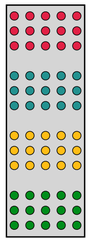
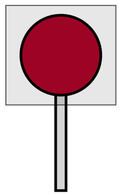


100 identical bowls of candy

1	2	3	4	5
				

Let u_i be the number of times candy i appears in Freddy's bag.

$u_1 = 17$ $u_2 = 26$ $u_3 = 13$ $u_4 = 21$ $u_5 = 23$

Let x_i be the number of times candy i appears in the bowl.

Question. Suppose that

$$u_1 = 17 \qquad u_2 = 26 \qquad u_3 = 13 \qquad u_4 = 21 \qquad u_5 = 23$$

What are the most likely values of x_1, x_2, x_3, x_4, x_5?

Answer : $\dfrac{x_i}{x_1 + x_2 + x_3 + x_4 + x_5} = \dfrac{u_i}{100}$ for $i = 1,\ldots,5$.

Harder Question. What are the most likely values of x_i assuming that x is in some algebraic variety $\mathcal{X}$?

Answer : Maximize the likelihood that u was sampled from x, subject to the constraint $x \in \mathcal{X}$.

Let x_i be the number of times that candy i appears in the bowl.

The probability of choosing candy i $\dfrac{x_i}{x_1 + x_2 + x_3 + x_4 + x_5}$.

Let x_i be the number of times that candy i appears in the bowl.

The probability of choosing candy i and candy j is

$$\frac{x_i}{x_1 + x_2 + x_3 + x_4 + x_5} \cdot \frac{x_j}{x_1 + x_2 + x_3 + x_4 + x_5}.$$

Let x_i be the number of times that candy i appears in the bowl and let u_i be the number of times candy i appears in the bag.

The probability of choosing the ith candy u_i times for $i = 1,2,3,4,5$ is (up to a constant)

$$L_u(x) = \frac{x_1^{u_1} \cdot x_2^{u_2} \cdot x_3^{u_3} \cdot x_4^{u_4} \cdot x_5^{u_5}}{(x_1 + x_2 + x_3 + x_4 + x_5)^{u_1+u_2+u_3+u_4+u_5}}$$

(likelihood function)

Answer : Maximize the likelihood that u was sampled from x, subject to the constraint $x \in \mathcal{X}$.

$$\text{maximize } L_u(x) \text{ subject to } x \in \mathcal{X} \qquad (\star)$$

Solve by:

1. Find all complex critical points of ($\star$).
2. Evaluate $L_u(x)$ at each real critical point to find the maximum.

Theorem (Huh-Sturmfels, 2014). For the generic u, the number of critical points of this optimization problem is constant and finite. This is called the *maximum likelihood (ML) degree.*

The maximum likelihood degree is an algebraic measure of complexity of ($\star$).

Theorem (Huh-Sturmfels, 2014). The maximum likelihood degree of a smooth d -dimensional variety $\mathscr{X} \subseteq \mathbb{P}^{n-1}$ is equal to the signed Euler characteristic $(-1)^d \chi(\mathscr{X} \backslash \mathscr{H})$.

$$\mathscr{H} = \{x \in \mathbb{P}^{n-1} : x_1 \cdots x_n (x_1 + \cdots + x_n) = 0\}$$

$$L_u(x) = \frac{x_1^{u_1} \cdot x_2^{u_2} \cdot x_3^{u_3} \cdot x_4^{u_4} \cdots x_n^{u_n}}{(x_1 + x_2 + x_3 + x_4 + \cdots + x_n)^{u_1+u_2+u_3+u_4+\cdots+u_n}}$$

The *Grassmannian* $\mathrm{Gr}(k,n)$ is the set of k-dimensional vector spaces in $\mathbb{C}^n$. It is realized as a (smooth) algebraic variety via the *Plücker embedding*.

Let A be a $k \times n$ matrix: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix}$

The *Plücker coordinates* are the maximal minors of A: $x_{i_1 i_2 \cdots i_k} = \begin{vmatrix} a_{1 i_1} & \cdots & a_{1 i_k} \\ \vdots & \ddots & \vdots \\ a_{k i_1} & \cdots & a_{k i_k} \end{vmatrix}$

Example $(k = 2,\ n = 5)$**.**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{pmatrix}$$

$$x_{ij} = \begin{vmatrix} a_{i1} a_{i2} \\ a_{j1} a_{j2} \end{vmatrix} = a_{i1} a_{j2} - a_{j1} a_{i2}$$

$$\begin{aligned} x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} &= 0 \\ x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23} &= 0 \\ x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24} &= 0 \\ x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34} &= 0 \\ x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} &= 0 \end{aligned}$$

Statistical Models on the Grassmannian:

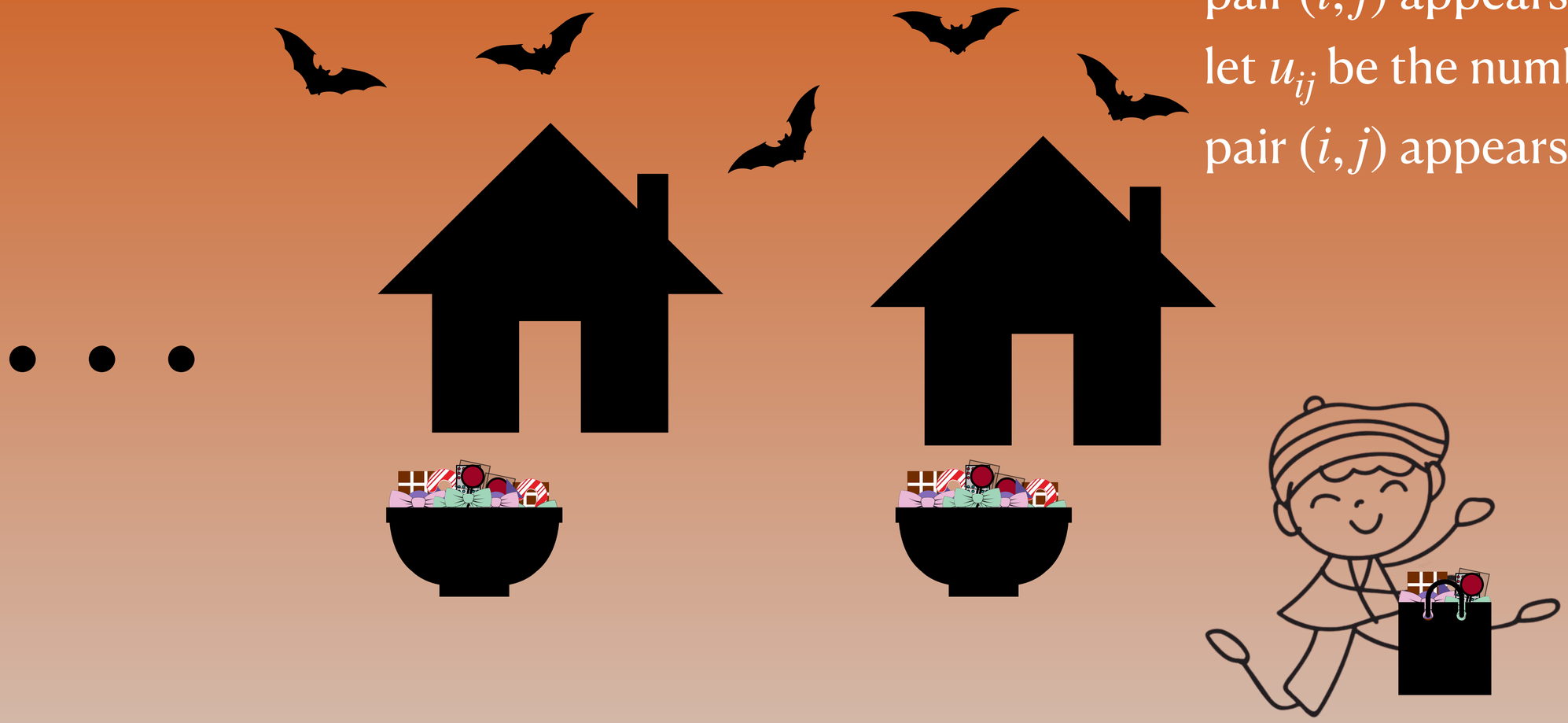
1. The positive Grassmannian
   $\mathrm{Gr}(k,n)_{>0} = \{x \in \mathrm{Gr}(k,n)_{\mathbb{R}} : \mathrm{sgn}(x_I) > 0 \text{ for all } I \subseteq [n], |I| = k\}.$
2. The configuration space $X(k,n) = \mathrm{Gr}(k,n)_{>0}/(\mathbb{R}_{>0})^n$.
3. The squared Grassmannian $\mathrm{sGr}(k,n) = \{(x_I^2)_I : x \in \mathrm{Gr}(k,n)_{\mathbb{R}}\}$.

Statistical Models on the Grassmannian of lines:

1. The positive Grassmannian
   $\mathrm{Gr}(2,n)_{>0} = \{x \in \mathrm{Gr}(2,n)_{\mathbb{R}} : \mathrm{sgn}(x_{ij}) > 0 \text{ for all } 1 \leq i < j \leq n\}$.
2. The configuration space $\mathscr{M}_{0,n} = X(2,n) = \mathrm{Gr}(2,n)_{>0}/(\mathbb{R}_{>0})^n$.
3. The squared Grassmannian $\mathrm{sGr}(2,n) = \{(x_{ij}^2)_{1\leq i<j\leq n} : x \in \mathrm{Gr}(2,n)_{\mathbb{R}}\}$.

Every point in these sets defines a probability distribution
on the set of 2-element subsets of $[n] = \{1,\ldots,n\}$.

Let x_{ij} be the number of times the pair (i,j) appears in the bowl and let u_{ij} be the number of times the pair (i,j) appears in the bag.



$$L_u(x) = \frac{x_{12}^{u_{12}}\, x_{13}^{u_{13}}\, x_{14}^{u_{14}}\, x_{15}^{u_{15}}\, x_{23}^{u_{23}}\, x_{24}^{u_{24}}\, x_{25}^{u_{25}}\, x_{34}^{u_{34}}\, x_{35}^{u_{35}}\, x_{45}^{u_{45}}}{(x_{12}+x_{13}+\cdots+x_{35}+x_{45})^{u_{12}+u_{13}+\cdots+u_{35}+u_{45}}}$$

(likelihood function)

		ML Degree (no. critical points)
maximize $L_u(x)$ subject to	$x \in \mathrm{Gr}(2,5)_{>0}$	22
	$x \in \mathscr{M}_{0,5}$	2
	$x \in \mathrm{sGr}(2,5)$	12

ML Degree (no. critical points)

maximize $L_u(x)$ subject to

$x \in \mathrm{Gr}(2,n)_{>0}$	??
$x \in \mathscr{M}_{0,n}$	$(n-3)!$ (Sturmfels-Telen, 2021)
$x \in \mathrm{sGr}(2,n)$	$(n-1)!/2$ (F, 2025)

Finally, $G(m, n)$ is the Plücker ideal of the Grassmannian of m-planes in $\mathbf{C}^n$.

Model	3×3	$3 \times 3sym$	3×4	$3 \times 3coin$	$3 \times 4coin$	$G(2, 4)$	$G(2, 5)$
unscaled	10	6	26	12	39	4	22
scaled	39	16	164	16	54	6	52

We close with two open problems, aimed at experts in enumerative geometry.

Problem 13. Find an explanation for all the ML degrees stated above.

(Hoşten-Khetan-Sturmfels, 2005)

Theorem (F, 2026). The ML degree of the Grassmannian Gr(2,n) is $(2^{n-1} - n)(n-3)!$.

Volume of the second hypersimplex

ML degree of $\mathscr{M}_{0,n}$

Proof.

Let $f(x) = \sum_{1 \le i < j \le n} x_{ij}$. Recall that the ML degree is equal to the Euler characteristic:

$$\text{ML Degree}(\text{Gr}(2,n)) = \chi(\text{Gr}(2,n)^\circ \backslash \{f = 0\})$$

Here, $\text{Gr}(2,n)^\circ = \{x \in \text{Gr}(2,n) : \prod_{1 \le i < j \le n} x_{ij} \neq 0\}$ denotes the open Grassmannian.

By the definition of $\mathscr{M}_{0,n}$, there exists a surjective map

$$\varphi : \text{Gr}(2,n)^\circ \backslash \{f = 0\} \longrightarrow \mathscr{M}_{0,n} = \text{Gr}(2,n)^\circ / (\mathbb{C}^\times)^n .$$

Assuming the fibers φ are also homeomorphic to a generic fiber $\mathscr{F}$, the multipliciativity of the of the Euler characteristic implies,

$$\chi(\text{Gr}(2,n)^\circ \backslash \{f = 0\}) = \chi(\mathscr{M}_{0,n}) \cdot \chi(\mathscr{F}) = (-1)^{n-1}(n-3)! \cdot \chi(\mathscr{F}) .$$

By the definition of $\mathscr{M}_{0,n}$, there exists a surjective map

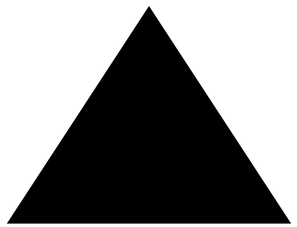
$$\varphi : \mathrm{Gr}(2,n)^\circ \backslash \{f = 0\} \longrightarrow \mathscr{M}_{0,n} = \mathrm{Gr}(2,n)^\circ / (\mathbb{C}^\times)^n .$$

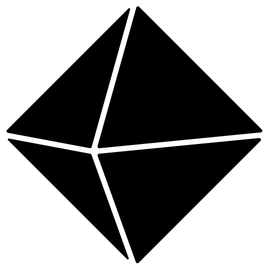
The fibers of φ are

$$\varphi^{-1}(x) \sim \{\lambda \in (\mathbb{C}^\times)^n : f_x(\lambda) \neq 0\}$$

where $f_x(\lambda) = \sum_{1 \leq i < j \leq n} x_{ij} \lambda_i \lambda_j$ is a polynomial in λ.

The fiber $\varphi^{-1}(x)$is the complement of a hypersurface whose Newton polytope is the second hypersimplex, and whose coefficients are in $\mathrm{Gr}(2,n)^\circ$.





The set $\{(\lambda \in (\mathbb{C}^\times)^n : f(\lambda \cdot x) \neq 0\}$ is the complement of a hypersurface whose Newton polytope is the second hypersimplex, and whose coefficients are in $\mathrm{Gr}(2,n)^\circ$.

Proposition (F, 2026). The principal A-determinant of the second hypersimplex $\Delta_{2,n}$ does not intersect the open Grassmannian $\mathrm{Gr}(2,n)^\circ$.

The set $\{(\lambda \in (\mathbb{C}^\times)^n : f(\lambda \cdot x) \neq 0\}$ is the complement of a **smooth** hypersurface whose Newton polytope is the second hypersimplex.

By (Kouchnirenko, 1967), the Euler characteristic of such a hypersurface complement is the signed volume of the second hypersimplex, namely

$$(-1)^{n-1}(2^{n-1} - n)\,.$$

$$\chi(\mathrm{Gr}(2,n)^\circ \setminus \{f = 0\}) = \chi(\mathscr{M}_{0,n}) \cdot \chi(\mathscr{F}) = (-1)^{n-1}(n-3)!(-1)^{n-1}(2^{n-1} - n)$$

Proposition (F, 2026). The principal A-determinant of the second hypersimplex $\Delta_{2,n}$ does not intersect the open Grassmannian $\mathrm{Gr}(2,n)^\circ$.

Proof. By (Clarke-Hoşten-Kushnerchuk-Oldekop, 2024), the principal A-determinant of $\Delta_{2,n}$ is the cut out by the product of principal minors of size at least 4 of the matrix

$$X = \begin{pmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{12} & 0 & x_{23} & \cdots & x_{2n} \\ x_{13} & x_{23} & 0 & \cdots & x_{3n} \\ \vdots & \vdots & \ddots & \vdots & \\ x_{1n} & x_{2n} & x_{3n} & \cdots & 0 \end{pmatrix}.$$

It suffices to show that if $x \in \mathrm{Gr}(2,n)$, the $\det(X) = 2^{n-2} x_{12}x_{23}\cdots x_{n-1,n}x_{1n}$.

To do this, we use a modified version of the *signed excedance enumerator*.

If the data are real and positive, how many critical points are real?

	ML Degree	Real	
$\mathrm{Gr}(2,n)_{>0}$	$(2^{n-1}-n)(n-3)!$	??	(F, 2026+)
$\mathscr{M}_{0,n}$	$(n-3)!$	$(n-3)!$	(Sturmfels-Telen, 2021)
$\mathrm{sGr}(2,n)$	$(n-1)!/2$	$(n-1)!/2$	(F, 2025)

Example.

The ML degree of Gr(2,4) is 4. Of these 4 complex critical points, either 2 or 4 of them are real. This behavior is governed by the logarithmic discriminant, a degree 14 polynomial with 8 427 terms:

$$
\begin{aligned}
&u_{12}^8u_{13}^2u_{14}^2u_{23}^2 + 2u_{12}^8u_{13}u_{14}^3u_{23}^2 + u_{12}^8u_{14}^4u_{23}^2 + 8u_{12}^7u_{13}^3u_{14}^2u_{23}^2 + 16u_{12}^7u_{13}^2u_{14}^3u_{23}^2\\
&+8u_{12}^7u_{13}u_{14}^4u_{23}^2 + 26u_{12}^6u_{13}^4u_{14}^2u_{23}^2 + 48u_{12}^6u_{13}^3u_{14}^3u_{23}^2 + 16u_{12}^6u_{13}^2u_{14}^4u_{23}^2 + 44u_{12}^5u_{13}^5u_{14}^2u_{23}^2\\
&+64u_{12}^5u_{13}^4u_{14}^3u_{23}^2 - 16u_{12}^5u_{13}^3u_{14}^4u_{23}^2 + 41u_{12}^4u_{13}^6u_{14}^2u_{23}^2 + 26u_{12}^4u_{13}^5u_{14}^3u_{23}^2 - 95u_{12}^4u_{13}^4u_{14}^4u_{23}^2\\
&+20u_{12}^3u_{13}^7u_{14}^2u_{23}^2 - 24u_{12}^3u_{13}^6u_{14}^3u_{23}^2 + 4u_{12}^2u_{13}^8u_{14}^2u_{23}^2 - 28u_{12}^2u_{13}^7u_{14}^3u_{23}^2 - 8u_{12}u_{13}^8u_{14}^3u_{23}^2 + \cdots
\end{aligned}
$$

If the data are real and positive, how many critical points are real and positive?

	ML Degree	Real	Positive	
$\mathrm{Gr}(2,n)_{>0}$	$(2^{n-1}-n)(n-3)!$	??	1	(F, 2026+)
$\mathscr{M}_{0,n}$	$(n-3)!$	$(n-3)!$	1	(Sturmfels-Telen, 2021)
$\mathrm{sGr}(2,n)$	$(n-1)!/2$	$(n-1)!/2$	$(n-1)!/2$	(F, 2025)

Theorem (F, 2026+). Given real generic data $u \in \mathbb{R}^{\binom{n}{2}}_{>0}$, the likelihood function $L_u(x)$ has one positive critical point.

Proof. Parameterize the positive Grassmannian $\mathrm{Gr}(2,n)_{>0}$ by

$$\begin{pmatrix} e^{r_1}\cos(\theta_1) & \cdots & e^{r_n}\cos(\theta_n) \\ e^{r_1}\sin(\theta_1) & \cdots & e^{r_n}\sin(\theta_n) \end{pmatrix}$$

where (r,θ) is in the convex set

$$\{r \in \mathbb{R}^n : r_0 = r_n = 0\} \times \{\theta \in [0,\pi/2]^n : 0 = \theta_1 < \theta_2 < \cdots < \theta_n = \pi/2\}\,.$$

Plücker coordinates: $\quad x_{ij} = e^{r_i+r_j}\sin(\theta_j - \theta_i)$

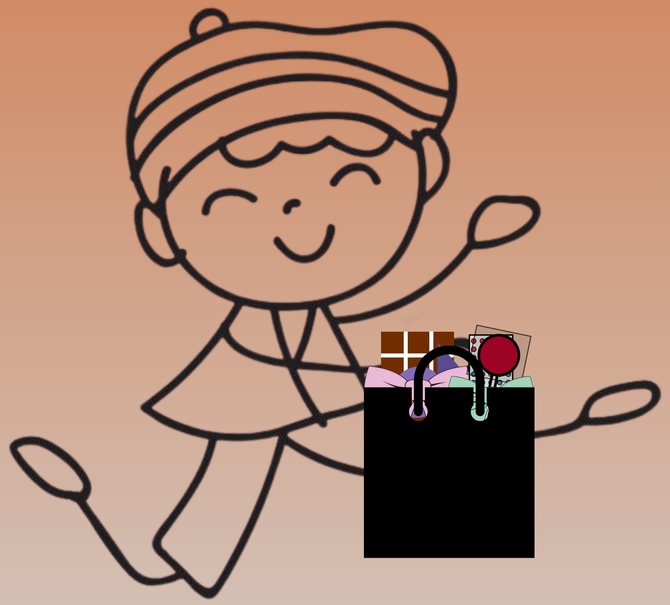
Argue that the log-likelihood function is concave on this set:

$$\sum_{1\le i<j\le n} u_{ij}(r_i + r_j + \log\sin(\theta_j-\theta_i)) \;+\; \Big(\sum_{1\le i<j\le n} u_{ij}\Big)\cdot\log\Big(\sum_{1\le i<j\le n} e^{r_i+r_j+\log\sin(\theta_j-\theta_i)}\Big)\,.$$

Theorem (F, 2026). The ML degree of the Grassmannian $\mathrm{Gr}(2,n)$ is $(2^{n-1}-n)(n-3)!$.

Theorem (F, 2026+). Given real generic data $u \in \mathbb{R}^{\binom{n}{2}}_{>0}$, the likelihood function $L_u(x)$ has one positive critical point.

Thank you!



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