Metric Algebraic Geometry of the Grassmannian of Lines

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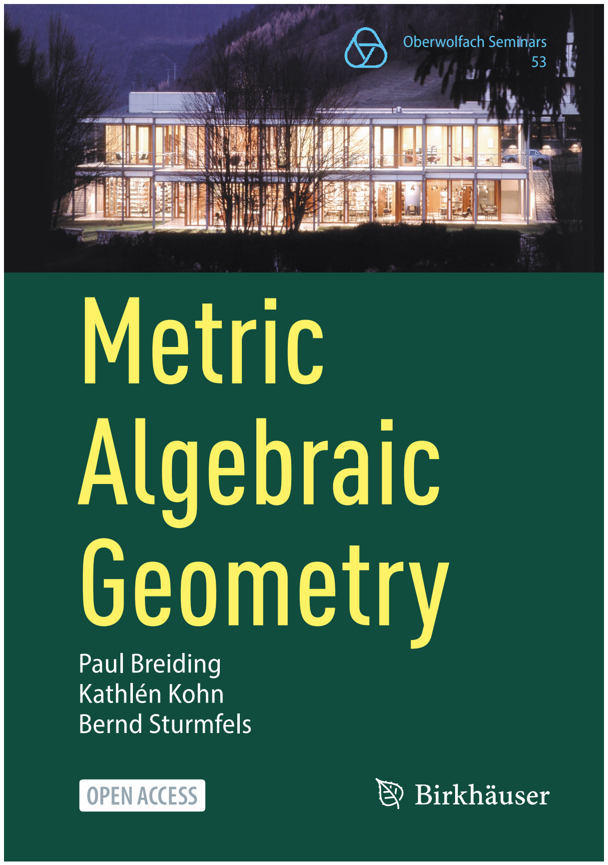
Based on joint work with Andrea Rosana and Bernd Sturmfels

Metric Algebraic Geometry



"…the interplay of metric concepts with algebraic objects."

Example.

How does one accurately compute the reach of an algebraic variety?

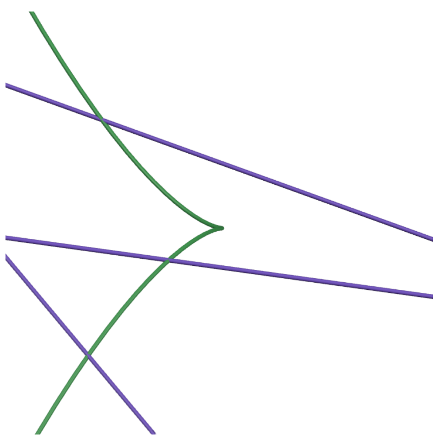
Today:

Euclidean distance optimization! (Chapter 2)

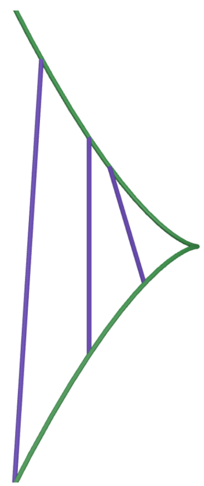
Models in the Grassmannian of Lines

The Grassmannian of lines is $\mathrm{Gr}(2,n) = \{2\text{-dimensional vector subspaces of } \mathbb{R}^n\}$.

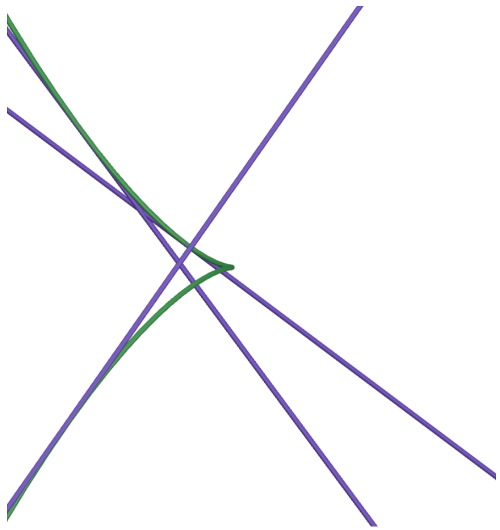
Chow Threefold



Secant Surface



Tangent Curve



Plücker Coordinates

L : 2-dimensional subspace of $\mathbb{R}^n$ $A : 2 \times n$ matrix whose rows span L

The *Plücker coordinates* for L are $x_{ij} = \det(A_{ij})$ for $1 \leq i < j \leq n$, where A_{ij} is the 2×2 submatrix of A formed by taking the columns indexed by i, j.

Example ($d = 2,\ n = 5$).

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{pmatrix}$$

$$x_{ij} = \begin{vmatrix} a_{i1} a_{i2} \\ a_{j1} a_{j2} \end{vmatrix} = a_{i1}a_{j2} - a_{j1}a_{i2}$$

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$$
$$x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23} = 0$$
$$x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24} = 0$$
$$x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34} = 0$$
$$x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} = 0$$

Projection Coordinates

L : 2-dimensional subspace of $\mathbb{R}^n$ A : $2 \times n$ matrix whose rows span L

The $n \times n$ matrix $P = A^T(AA^T)^{-1}A$ is the unique orthogonal projection matrix onto L.

$$\mathrm{pGr}(2,n) = \{P \in \mathrm{Sym}^2\mathbb{C}^n : P^2 = P, \mathrm{trace}(P) = 2\}$$

Theorem (Devriendt-F-Reinke-Sturmfels, 2024). The projection Grassmannian $\mathrm{pGr}(2,n)$ is a smooth, affine variety of dimension $2n-4$. It has prime ideal

$$I(\mathrm{pGr}(2,n)) = \langle P^2 - P, \mathrm{trace}(P) - 2 \rangle.$$

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{12} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{pmatrix}$$

The Grassmannian of Lines

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \end{pmatrix}$$

$$X = A^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1n} \\ -x_{12} & 0 & x_{23} & \cdots & x_{2n} \\ -x_{13} & -x_{23} & 0 & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{1n} & -x_{2n} & -x_{3n} & \cdots & 0 \end{pmatrix}$$

$$P = A^T(AA^T)^{-1}A = \frac{2}{\mathrm{trace}(X^2)} X^2 = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{12} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{pmatrix}$$

Example: Chow Threefold

Polynomials like

Parametrization: $$A = \begin{pmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & u & v \end{pmatrix}$$

Plücker Embedding: $$\det \begin{pmatrix} x_{12} & x_{13} & x_{23} \\ x_{13} & x_{14} + x_{23} & x_{24} \\ x_{23} & x_{24} & x_{34} \end{pmatrix}$$

Projection Coordinates:

$$\begin{aligned} &2p_{23}^2p_{24}p_{33} - 2p_{22}p_{24}p_{33}^2 - 2p_{23}^3p_{34} + 2p_{14}p_{22}p_{24}p_{34} - 2p_{13}p_{23}p_{24}p_{34} - 2p_{12}p_{24}^2p_{34} + 2p_{22}p_{23}p_{33}p_{34} \\ &+2p_{12}p_{24}p_{33}p_{34} - 4p_{13}p_{22}p_{34}^2 + 4p_{12}p_{23}p_{34}^2 - 2p_{23}^2p_{34}^2 + p_{11}p_{24}p_{34}^2 + p_{22}p_{24}p_{34}^2 - p_{11}p_{33}p_{34}^2 \\ &+p_{22}p_{33}p_{34}^2 - p_{24}p_{33}p_{34}^2 - p_{33}^2p_{34}^2 + 2p_{23}p_{34}^3 - 2p_{34}^4 - 2p_{14}p_{22}p_{23}p_{44} + 2p_{13}p_{23}^2p_{44} + 2p_{12}p_{23}p_{24}p_{44} \\ &+4p_{13}p_{22}p_{33}p_{44} - 6p_{12}p_{23}p_{33}p_{44} + 2p_{23}^2p_{33}p_{44} - p_{11}p_{24}p_{33}p_{44} - p_{22}p_{24}p_{33}p_{44} + p_{11}p_{33}^2p_{44} \\ &-p_{22}p_{33}^2p_{44} + p_{24}p_{33}^2p_{44} + p_{33}^3p_{44} - 2p_{23}p_{33}p_{34}p_{44} + p_{24}p_{34}^2p_{44} + 3p_{33}p_{34}^2p_{44} - p_{24}p_{33}p_{44}^2 - p_{33}^2p_{44}^2 \end{aligned}$$

Part 1: Squaring Skew Symmetric Matrices

What happens when we multiply two skew symmetric matrices?

H. Stenzel*: Über die Darstellbarkeit einer Matrix als Produkt von zwei symmetrischen Matrizen, als Produkt von zwei alternierenden Matrizen und als Produkt von einer symmetrischen und einer alternierenden Matrix,* Mathematische Zeitschrift **15** (1922) 1-25.

The Squaring Map

$$(\,-\,)^2 : \mathbb{P}(\wedge^2\mathbb{C}^n) \dashrightarrow \mathbb{P}(\mathrm{Sym}^2\mathbb{C}^n), \qquad X \mapsto X^2$$

Write $\mathscr{V}_{X^2}$ the Zariski closure of the image of $(\,-\,)^2$.

Theorem (F-Rosana-Sturmfels, 2026).

- The variety $\mathscr{V}_{X^2}$ has dimension $\binom{n}{2} - 1$.
- $\mathscr{V}_{X^2}$ is the Zariski closure of the set of symmetric diagonalizable matrices whose nonzero eigenvalues have multiplicity 2.
- The image of the real locus $\mathbb{P}(\wedge^2\mathbb{R}^n)$ consists of negative definite matrices $P \in \mathrm{Sym}^2\mathbb{R}^n$ whose nonzero eigenvalues have even multiplicity.
- If P has maximal rank, every complex matrix $X \in \wedge^2\,\mathbb{C}^n$ with $X^2 = P$ is real.

The Base Locus

The map $(\,-\,)^2$ commutes with conjugation by the orthogonal group

$$\mathrm{O}(n) = \{R \in \mathbb{C}^{n \times n} : RR^T = \mathrm{Id}\}\,.$$

Up to the $\mathrm{O}(n)$ action, the base locus consist of block diagonal matrices with blocks (0) and

$$\begin{pmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & i \\ -i & 0 & 0 & -1 \\ 0 & -i & 1 & 0 \end{pmatrix}.$$

Example: $n = 4$

$$X = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix} \qquad P = X^2 = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{22} & p_{23} & p_{24} \\ p_{13} & p_{23} & p_{33} & p_{34} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{pmatrix}$$

$\mathscr{V}_{X^2}$ has dimension 5 and degree 6 in the $\mathbb{P}^9$ of symmetric 4×4 matrices P. Its prime ideal is generated by 2×2 minors of

$$\begin{pmatrix} p_{11} - p_{22} - p_{33} + p_{44} & 2p_{13} + 2p_{24} & 2p_{12} - 2p_{34} \\ 2p_{13} - 2p_{24} & -p_{11} - p_{22} + p_{33} + p_{44} & 2p_{14} + 2p_{23} \\ 2p_{12} + 2p_{34} & -2p_{14} + 2p_{23} & -p_{11} + p_{22} - p_{33} + p_{44} \end{pmatrix}$$

The base locus has two components:

$$\langle x_{14} - x_{23}, x_{13} + x_{24}, x_{12} - x_{34}, x_{23}^2 + x_{24}^2 + x_{34}^2 \rangle \cap \langle x_{14} + x_{23}, x_{13} - x_{24}, x_{12} + x_{34}, x_{23}^2 + x_{24}^2 + x_{34}^2 \rangle$$

The Grassmannian

The Grassmannian $\mathrm{Gr}(2,n)$ in its Plücker embedding is the rank 2 locus of skew symmetric matrices.

The projection Grassmannian is the affine chart $\mathrm{trace}(P) \neq 0$ of the image under the squaring map $(\mathrm{Gr}(2,n))^2$.

Theorem (Lim-Ye, 2025). The projection Grassmannian $\mathrm{pGr}(2,n)$ has dimension $2n-4$ and degree $2\binom{2n-4}{n-2}$.

Schubert Varieties

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbb{R}^n$$

Write $\mathcal{S}_{ij}$ for the Schubert variety

$$\mathcal{S}_{ij} = \{L \in \mathrm{Gr}(2,n) : \dim(L \cap E_i) \geq 2, \dim(L \cap E_j) \geq 1\}\,.$$

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & * & \cdots & * \end{pmatrix}$$
$$\qquad\qquad\qquad i \qquad\qquad\qquad j$$

ideal of $\mathcal{S}_{ij}$ $=$ ideal of $\mathrm{Gr}(2,n)$ $+$ $\langle x_{rs} : r < i$ or $s < j\rangle$

Conjecture (F-Rosana-Sturmfels, 2026). The prime ideal of $\mathcal{S}^2_{ij}$ is generated by the entries of $2P^2 - \mathrm{trace}(P) \cdot P$, the 3×3 minors of P, and the 2×2 minors of the submatrices of P formed by rows r and s with $r < i$ or $s < j$.

Degrees of Schubert Varieties

Conjecture (F-Rosana-Sturmfels, 2026). For $j < n$, the varieties $\mathcal{S}^2_{ij}$ satisfy $\deg(\mathcal{S}_{i,n-1}) = 4(2^{n-i-1} - 1)$ and

$$\deg(\mathcal{S}^2_{ij}) = \deg(\mathcal{S}^2_{i+1,j}) + \deg(\mathcal{S}^2_{i,j+1})\,.$$

$n-j$ \ $n-i$	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	32	64	128	256
1		4	12	28	60	124	252	508	1020
2			12	40	100	224	476	984	2004
3				40	140	364	840	1824	3828
4					140	504	1344	3168	6996
5						504	1848	5016	12 012
6							1848	6864	18 876
7								6864	25 740

Part 2: Distance Optimization

Two Distance Optimization Perspectives

Let $\mathscr{M} \subseteq \mathbb{C}^n$ be an algebraic variety.

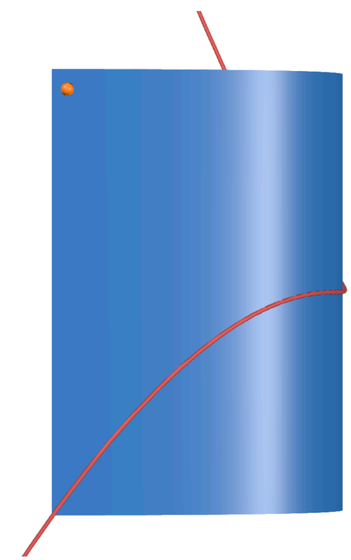
Given a data point $u \in \mathbb{R}^n$, what point on $\mathscr{M}_{\mathbb{R}}$ is closest to u?

Euclidean distance, Kullback-Leibler divergence, …

Let $\mathscr{V}$ be an algebraic variety, and let $\mathscr{M} \subseteq \mathscr{V}$ be a subvariety.

Given a data point $u \in \mathscr{V}_{\mathbb{R}}$, what point on $\mathscr{M}_{\mathbb{R}}$ is closest to u?

Euclidean distance, geodesics, …



The Euclidean Distance Degree

Given $f_1, \ldots, f_r \in \mathbb{R}[P]$, define a model $\mathscr{M} = V(f_1, \ldots, f_r) \cap \mathrm{pGr}(2,n) \subseteq \mathbb{C}^{\binom{n+1}{2}}$.

Given a data point $Q \in \mathrm{Sym}^2\mathbb{R}^n$, what point on $\mathscr{M}$ is closest to u?

$$\min_{P \in \mathscr{M}_{\mathbb{R}}} ||P - Q||_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (p_{ij} - q_{ij})^2$$

Critical points: $\{P \in \mathscr{M}_{\mathrm{reg}} : \mathrm{rank}(\mathscr{J}^Q) = \mathrm{codim}(\mathscr{M})\}$

$$\mathscr{J}^Q = \begin{pmatrix} \nabla ||P - Q||_F^2 \\ \mathrm{Jac}_{\mathrm{Gr}(2,n)}(P) \\ -\nabla f_1(x)- \\ \vdots \\ -\nabla f_r(x)- \end{pmatrix}$$

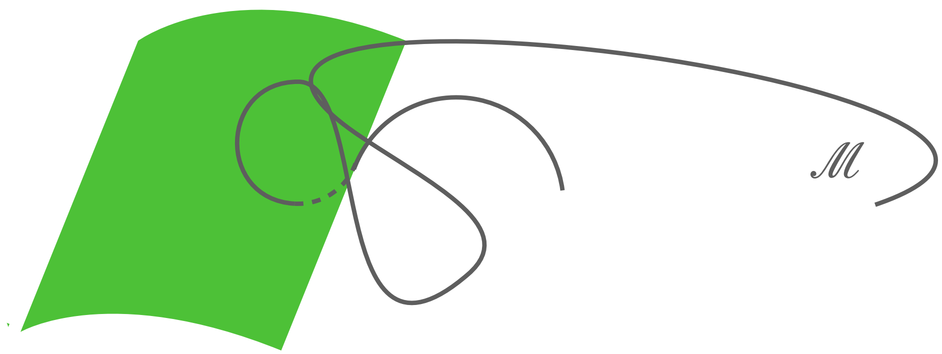
For generic data $Q \in \mathrm{Sym}^2\mathbb{R}^n$, this set is zero dimensional. The Euclidean distance degree (ED degree) of $\mathscr{M}$ is its cardinality.

Extraneous Critical Points

Given $f_1, \ldots, f_r \in \mathbb{R}[P]$, define a model $\mathscr{M} = V(f_1, \ldots, f_r) \cap \mathrm{pGr}(2,n) \subseteq \mathbb{C}^{\binom{n+1}{2}}$.

Suppose now that $Q \in \mathrm{pGr}(2,n)$.

$$\{P \in \mathscr{M}_{\mathrm{reg}} : \mathrm{rank}(\mathscr{J}^Q) = \mathrm{codim}(\mathscr{M})\}$$



$$\mathscr{J}^Q = \begin{pmatrix} \nabla \| P - Q \|_F^2 \\ \mathrm{Jac}_{\mathrm{Gr}(2,n)}(P) \\ -\nabla f_1(x)- \\ \vdots \\ -\nabla f_r(x)- \end{pmatrix}$$

Extraneous critical points are points $P \in \mathrm{pGr}(2,n)$ at which $\mathscr{J}^Q$ drops rank no matter what the model is.

The Grassmann Distance Degree

Proposition (F-Rosana-Sturmfels, 2026). The extraneous critical points are the Q-invariant subspaces of dimension 2. These are the points P satisfying $PQ = QP$.

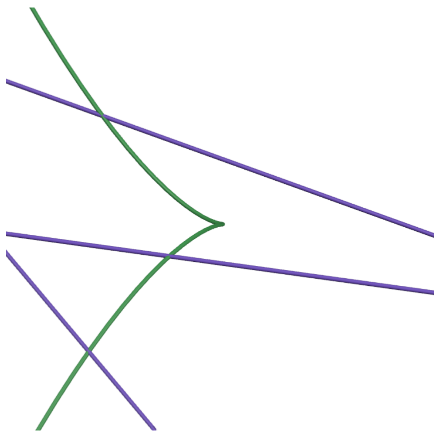
For most models, and generic $Q \in \mathrm{pGr}(2,n)$, the critical point set has finitely many points after removing the extraneous critical points. The Grassmann distance degree (GD degree) is the cardinality of

$$\{P \in \mathscr{M}_{\mathrm{reg}} : \mathrm{rank}(\mathscr{J}^u) = \mathrm{codim}(\mathscr{M}), PQ \neq QP\}$$

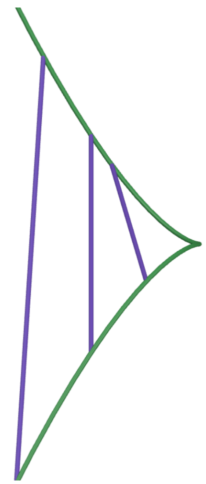
Theorem (F-Rosana-Sturmfels, 2026). For $n \geq 6$ and a model $\mathscr{M}$ of dimension at least 4, the ED degree of $\mathscr{M}$ is strictly larger than the GD degree of $\mathscr{M}$.

Lines in $\mathbb{P}^3$

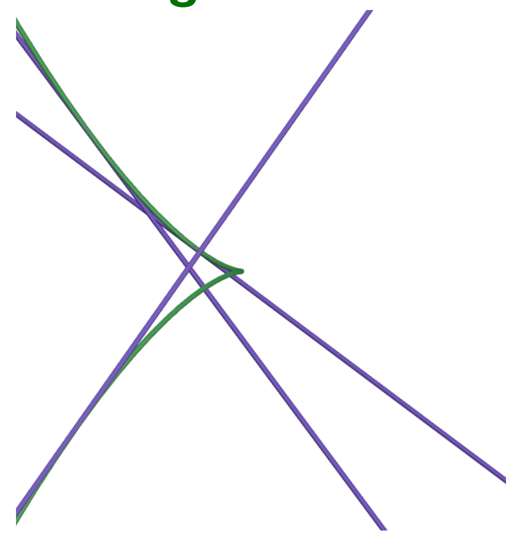
Chow Threefold



Secant Surface



Tangent Curve



Dimension	3	2	1
ED Degree	42	19	14
GD Degree	10	15	14

GD Degrees of Schubert Varieties

Write $\mathscr{S}_{ij}$ for the Schubert variety

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & * & \cdots & * \end{pmatrix}$$
$$\qquad\qquad\qquad i \qquad\qquad\qquad\quad j$$

Theorem (F-Rosana-Sturmfels, 2026). The Schubert varieties $\mathscr{S}_{1n}$ and $\mathscr{S}_{ij}$ with $j - i = 1$ have GD degree 1.

Conjecture (F-Rosana-Sturmfels, 2026). All other Schubert varieties in $\mathrm{Gr}(2,n)$ have GD degree 2.

Part 3: The Spectral Region

Measuring Distance

Theorem (Ye-Lim, 2016). Any nice (rotation-invariant) distance between two subspaces in $\mathrm{Gr}(2,n)$ depends only on the *principal angles* between the subspaces.

Principal Angles:

If P and Q are symmetric projection matrices onto 2-dimensional subspaces of $\mathbb{R}^n$, then the matrix PQ has 2 positive eigenvalues $\lambda \geq \mu$.

The principal angles are $\theta_1 = \cos^{-1}(\sqrt{\lambda})$ and $\theta_2 = \cos^{-1}(\sqrt{\mu})$.

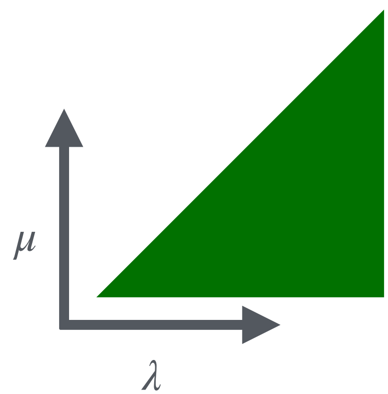
Upshot: the goal of any distance minimization problem is to simultaneously minimize the principal angles.

Measuring Distance

Given a model $\mathscr{M} \subseteq \mathrm{Gr}(2,n)$ and a data point $Q \in \mathrm{pGr}(2,n)$, there is a function

$$\mathscr{M}_{\mathbb{R}} \to \mathbb{R}^2 \qquad P \mapsto (\lambda, \mu)$$

where $1 \geq \lambda \geq \mu > 0$ are the nonzero eigenvalues of PQ. The image of this map is the *spectral region*. This is a 2-dimensional subset of the semi algebraic set



Example: $\mathcal{S}_{25}$

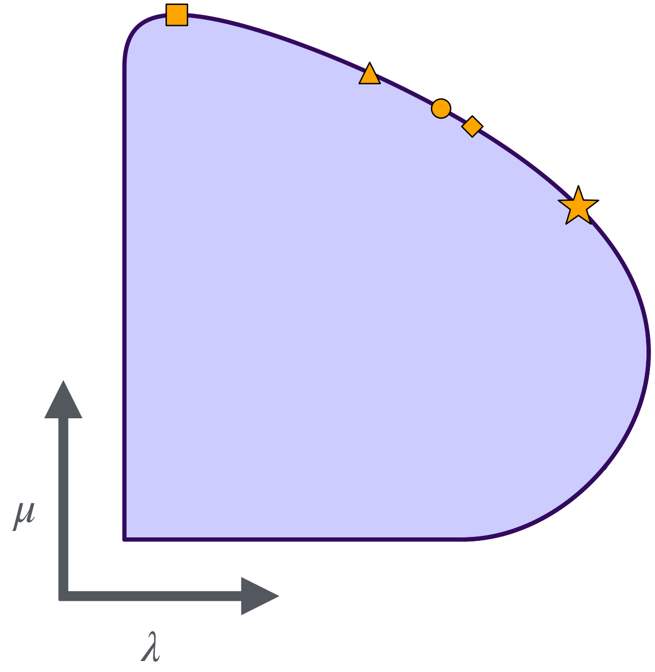
$$A = \begin{pmatrix} 0 & x & y & z & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \frac{1}{x^2+y^2+z^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & x^2 & xy & xz & 0 \\ 0 & xy & y^2 & yz & 0 \\ 0 & xz & yz & z^2 & 0 \\ 0 & 0 & 0 & 0 & x^2+y^2+z^2 \end{pmatrix}$$

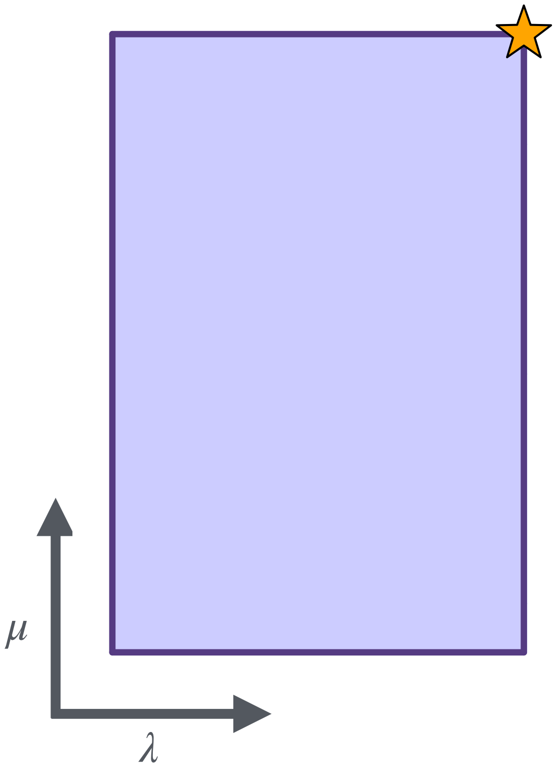
$$Q = \frac{1}{10} \begin{pmatrix} 6 & 4 & 2 & 0 & -2 \\ 4 & 3 & 2 & 1 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ -2 & 0 & 2 & 3 & 6 \end{pmatrix}$$

The Spectral Region

	Distance squared in λ, μ	Minimizer
Chordal	$4 - 2\lambda - 2\mu$	☆
Geodesic	$\arccos(\sqrt{\lambda})^2 + \arccos(\sqrt{\mu})^2$	○
Procrustes	$4 - 2\sqrt{\lambda} - 2\sqrt{\mu}$	◇
Binet-Cauchy	$1 - \lambda\mu$	△
Fubini-Study	$\arccos(\sqrt{\lambda\mu})^2$	△
Martin	$-\log(\lambda\mu)$	△
Asimov	$\arccos(\sqrt{\mu})^2$	□
Projection	$1 - \mu$	□
Spectral	$2 - 2\sqrt{\mu}$	□



Snugness



All distances have the same maximal eigenvalues!

Models with this property are called *snug*.

Theorem (F-Rosana-Sturmfels, 2026). The Schubert varieties $\mathcal{S}_{ij}$ with $i = 1$ or $j - i = 1$ are snug.

Thank you!

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- H. Friedman, A. Rosana, B. Sturmfels: *Distance optimization in the Grassmannian of lines*, 2026.
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