Counting Homogeneous Einstein Metrics

Hannah Friedman (UC Berkeley)

San Francisco State University Algebra, Geometry, and Combinatorics Seminar

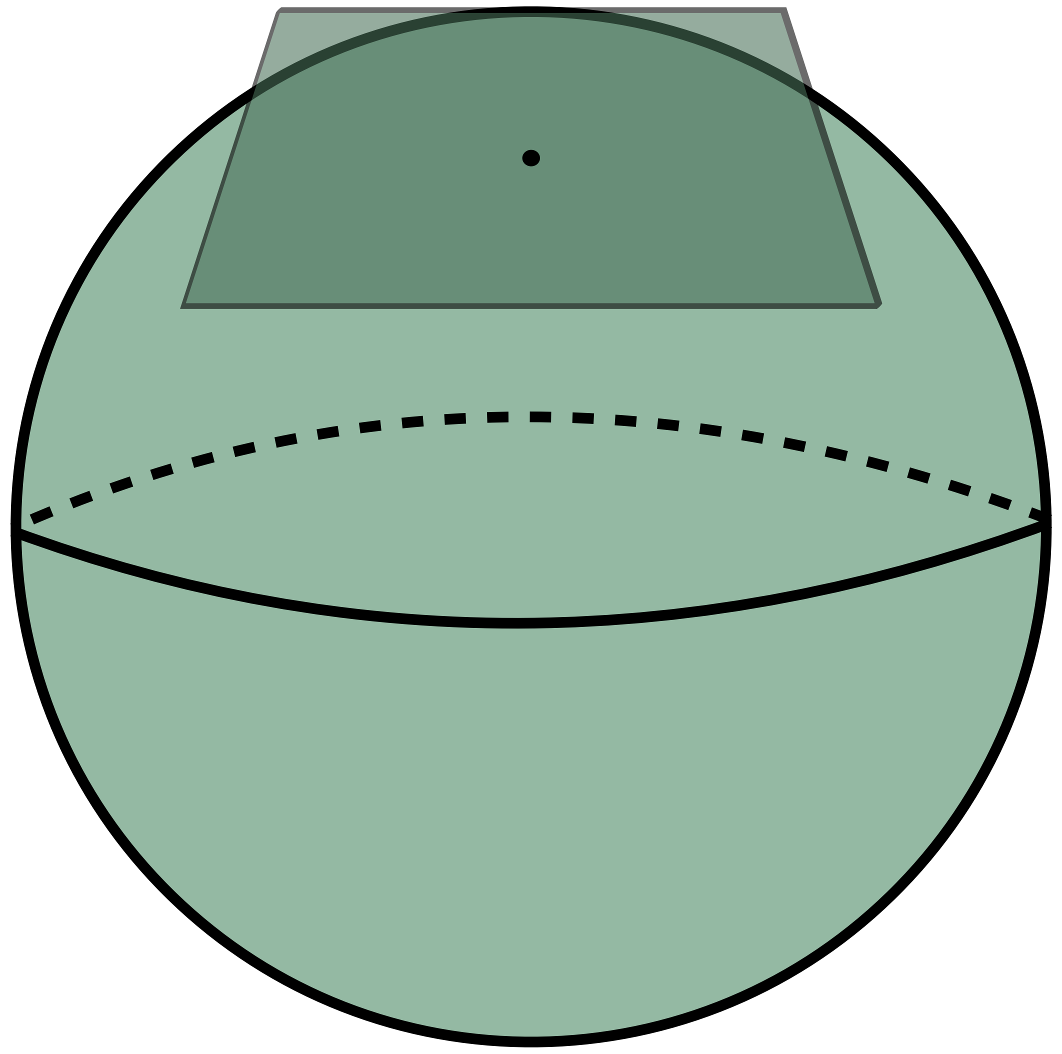
October 8, 2025

Joint work with Renato G. Bettiol

(Homogeneous) Einstein Metrics

Let M be a manifold with metric g. The metric g is *Einstein* if $\mathrm{Ric}_g = \lambda g$.

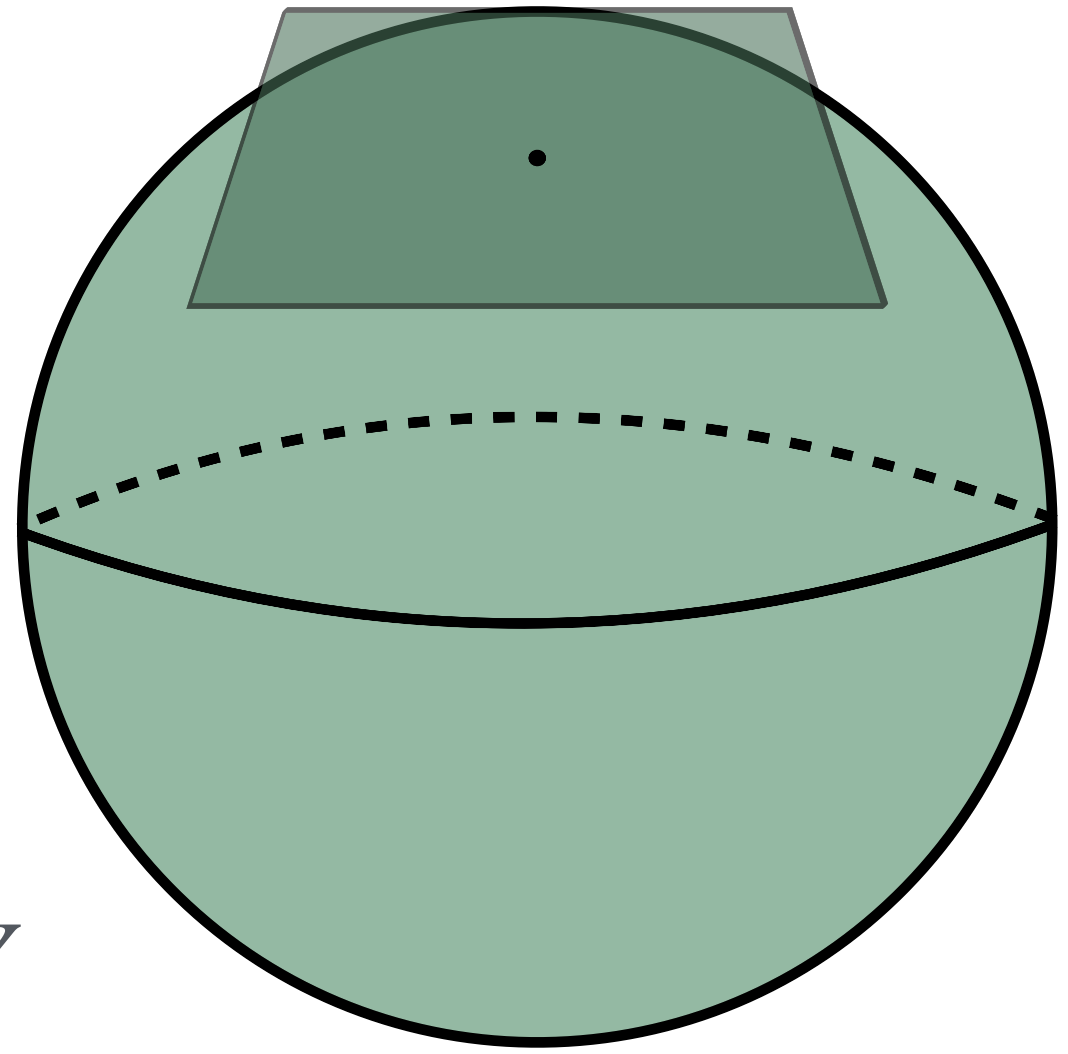
If a group G acts on M in a way that respects g, then the metric g is uniquely determined by an inner product on the tangent space $\mathfrak{m}$ of M at a single point. Assume $\mathfrak{m}$ decomposes into pairwise inequivalent Q-orthogonal representations $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\ell$. Then $g = x_1 Q|_{\mathfrak{m}_1} + \cdots + x_\ell Q|_{\mathfrak{m}_\ell}$ and g is Einstein if and only if



$$f_i(x) = \frac{L_{iii}}{x_i} + \sum_{k \in [\ell] \setminus \{i\}} \left(\frac{2L_{iik}x_k}{x_i^2} - \frac{2L_{ikk}x_i}{x_k^2} \right) + \sum_{j \neq k \in [\ell] \setminus \{i\}} 2L_{ijk} \left(\frac{x_j}{x_i x_k} + \frac{x_k}{x_i x_j} - \frac{x_i}{x_j x_k} \right) + 4d_i = 0, \quad i \in [\ell]$$

$d_i = \dim(\mathfrak{m}_i)$ $\qquad$ L_{ijk} are (almost) structure constants

(Homogeneous) Einstein Metrics



Example (Ziller, 1982).

$M = S^{2n+1} \subseteq \mathbb{C}^{n+1}$ $\qquad G = \mathrm{SU}(n+1)$

$M \cong \mathrm{G/H}$ where $\mathrm{H} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{G} : A \in \mathrm{SU}(n) \right\}$

endow $\mathfrak{g} = \mathfrak{su}(n+1)$ with the bi-invariant metric $Q(X,Y) = -\frac{1}{2}\mathrm{Re}\,\mathrm{tr}XY$

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ where $\mathfrak{m}_1 \cong \mathbb{C}^n$ is the defining representation and $\mathfrak{m}_2 \cong \mathbb{R}$ is the trivial representation

$$\ell = 2 \quad d_1 = 2n \quad d_2 = 1 \quad L_{112} = 4n+4 \quad L_{122} = 0 \quad L_{111} = -16n^2 + 16n \quad L_{222} = 0$$

$$\begin{aligned} -\frac{16n^2+16n}{x_1} + \frac{(8n+8)x_2}{x_1^2} &= -8n \\ \frac{(4n+4)x_2}{x_1^2} &= 4 \end{aligned} \implies x_1 = 2n, \quad x_2 = \frac{2n}{n+1}$$

round metric

The Finiteness Conjecture

Conjecture (Böhm-Wang-Ziller, 2004). If $M = \mathsf{G/H}$ is a compact homogeneous space whose isotropy representation consists of pairwise inequivalent irreducible summands, then the Einstein equations have only finitely many real solutions.

$$f_i(x) = \frac{L_{iii}}{x_i} + \sum_{k\in[\ell]\setminus\{i\}} \left(\frac{2L_{iik}x_k}{x_i^2} - \frac{2L_{ikk}x_i}{x_k^2} \right) + \sum_{j\neq k\in[\ell]\setminus\{i\}} 2L_{ijk} \left(\frac{x_j}{x_i x_k} + \frac{x_k}{x_i x_j} - \frac{x_i}{x_j x_k} \right) + 4d_i = 0, \quad i \in [\ell]$$

Proposition (Bettiol-F, 2025). Let $\ell = 2$ and suppose that $d_1, d_2 > 0$. Then the system has finitely many solutions.

Question (Bettiol-F, 2025). Suppose $d_1, \ldots, d_\ell > 0$. Does this system have finitely many solutions?

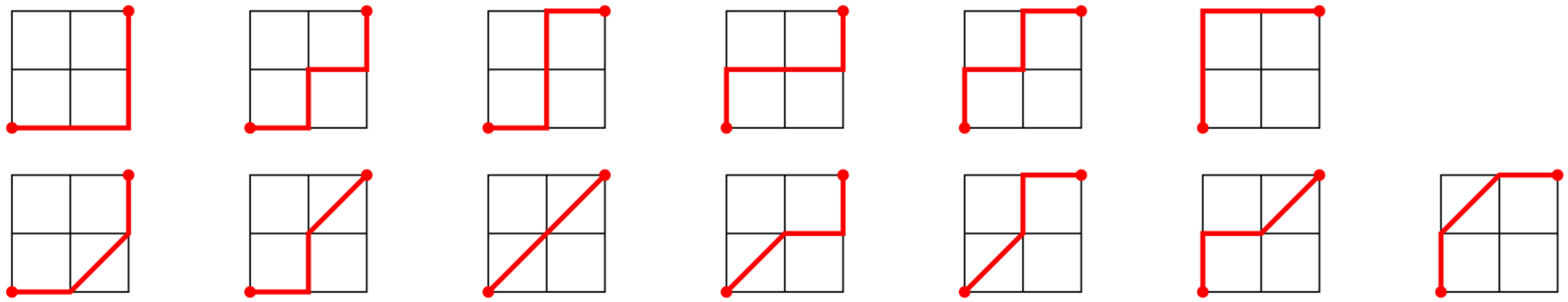
An Upper Bound

Theorem (Bettiol-F, 2025). The number of isolated solutions in $(\mathbb{C}^\times)^\ell$ to

$$f_i(x) = \frac{L_{iii}}{x_i} + \sum_{k\in[\ell]\setminus\{i\}} \left(\frac{2L_{iik}x_k}{x_i^2} - \frac{L_{ikk}x_i}{x_k^2} \right) + \sum_{j\neq k\in[\ell]\setminus\{i\}} 2L_{ijk} \left(\frac{x_j}{x_i x_k} + \frac{x_k}{x_i x_j} - \frac{x_i}{x_j x_k} \right) + 4d_i = 0, \quad i \in [\ell]$$

is at most the central Delannoy number $D_{\ell-1}$.

$D_2 = 13$:

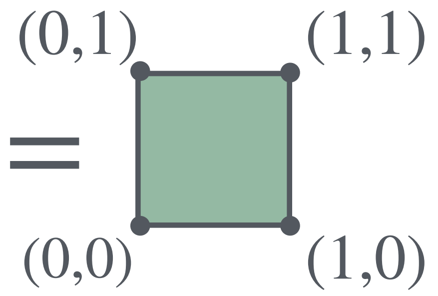


3, 13, 63, 321, 1683, 8989, 48 639, 265 729

The Bernstein-Khovanskii-Kushnirenko Bound

A Laurent polynomial $f(x) = \sum_{a \in \mathbb{Z}^\ell} c_a \underbrace{x^a}_{x_1^{a_1}\cdots x_\ell^{a_\ell}}$ has support $\mathscr{A} = \{a \in \mathbb{Z}^\ell : c_a \neq 0\}$ and Newton polytope $P = \mathrm{conv}(\mathscr{A})$.

$f(x, y) = 1 + x + y + xy$ $\qquad \mathscr{A} = \{(0,0), (1,0), (0,1), (1,1)\}$ $\qquad P =$



Let $\mathscr{F} = \{f_1, \ldots, f_\ell\}$ be a system of Laurent polynomials with Newton polytopes $\mathbf{Newt}(f_i) = P_i$.

Theorem (Bernstein, 1975). The number of isolated solutions in $(\mathbb{C}^\times)^\ell$ to system $\mathscr{F} = 0$ is bounded above by the *mixed volume* $\mathrm{MV}(P_1, \ldots, P_\ell)$.

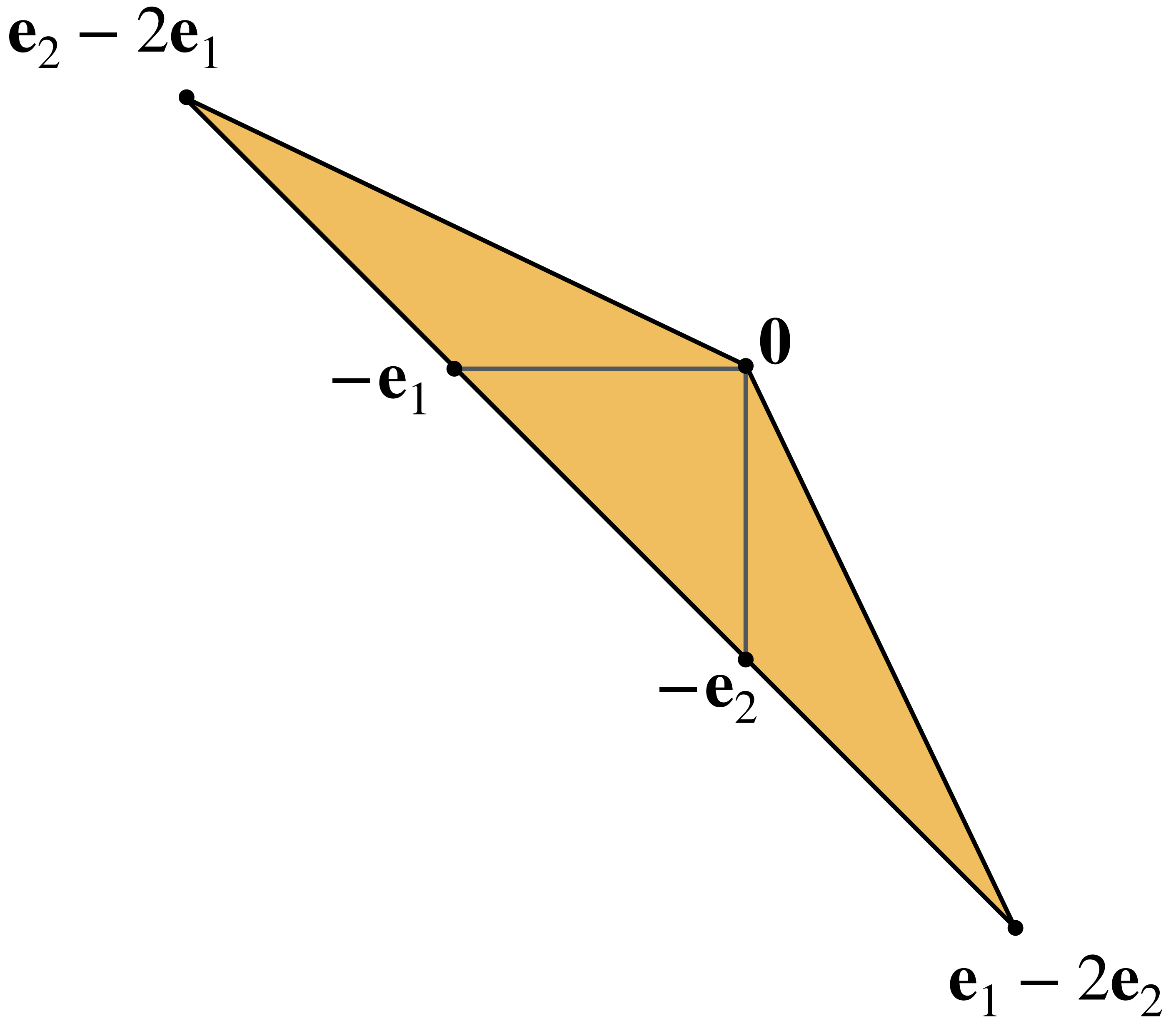
If $P_1, \ldots, P_\ell$ are "close enough," then $\mathrm{MV}(P_1, \ldots, P_\ell) = \mathrm{nVol}(P_1 \cup \cdots \cup P_\ell)$.

An Upper Bound: Examples

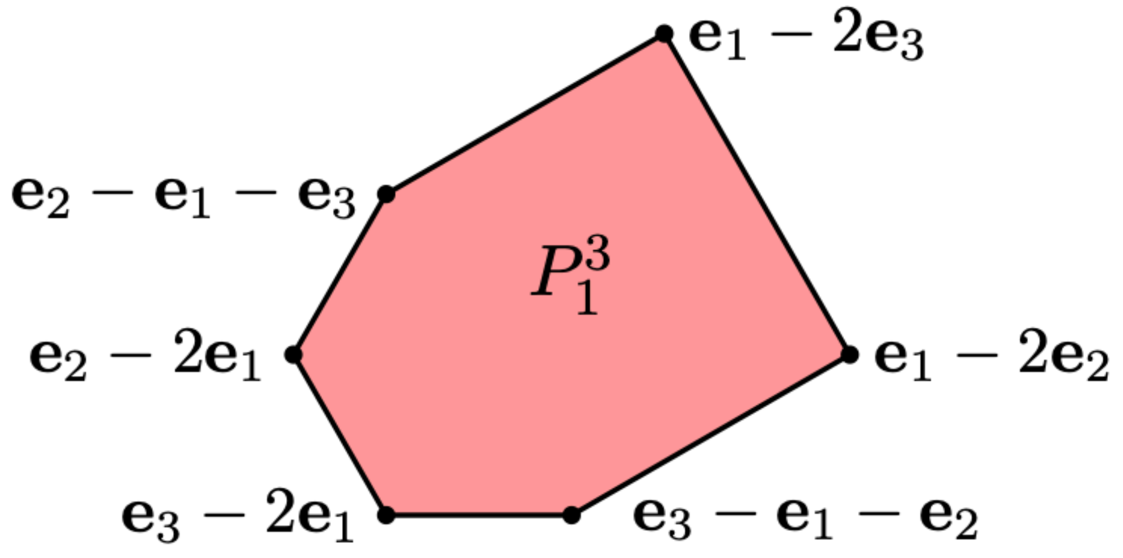
$\ell = 2:$

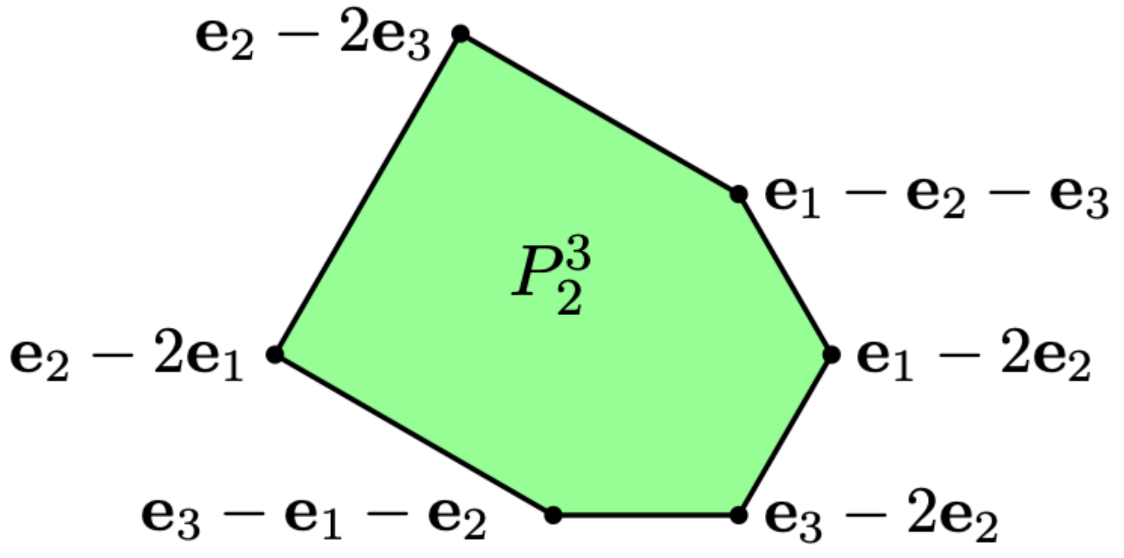
$$f_1(x) = L_{111}x_1^{-1} + 2L_{112}x_2x_1^{-2} - L_{122}x_1x_2^{-2} + 4d_1 = 0$$

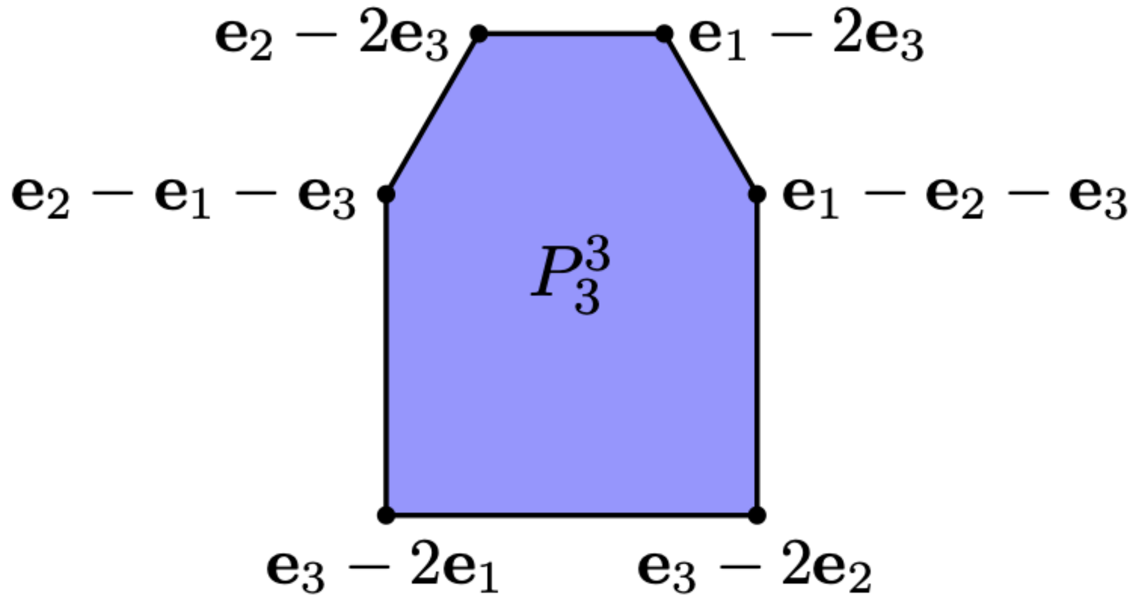
$$f_2(x) = L_{222}x_2^{-1} + 2L_{122}x_1x_2^{-2} - L_{112}x_2x_1^{-2} + 4d_2 = 0$$

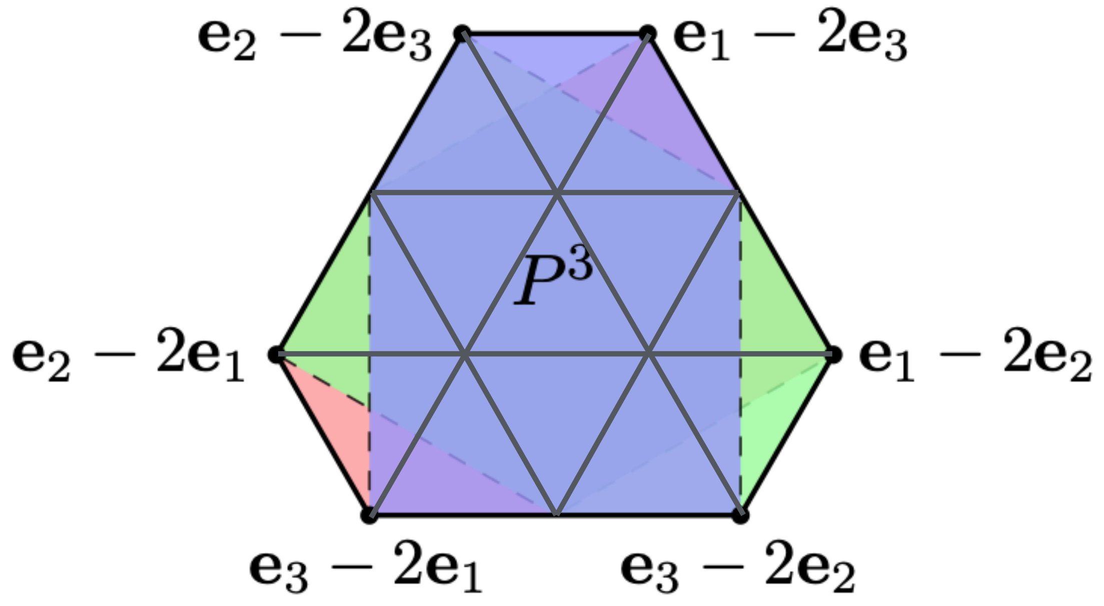


$\ell = 3:$









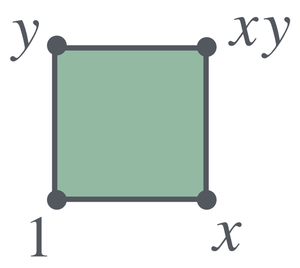
$$P = \operatorname{conv}(\mathbf{0}, \mathbf{e}_i - 2\mathbf{e}_j : i, j \in [\ell])$$

Bernstein's "Other" Theorem

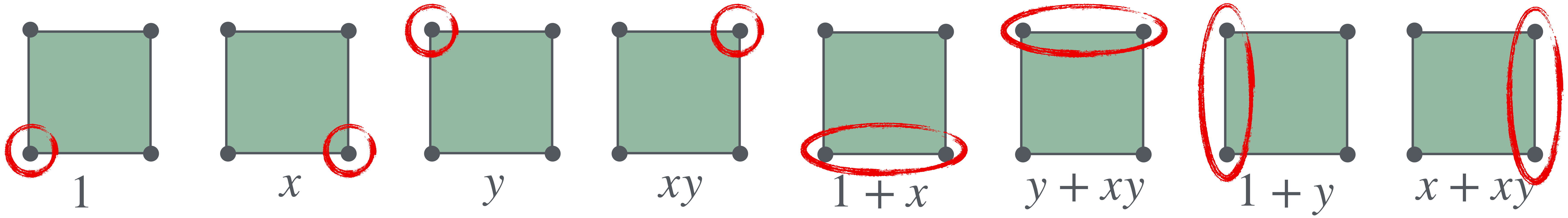
Theorem (Bernstein, 1975). Suppose that every facial system of $\mathscr{F}$ has no roots in $(\mathbb{C}^{\times})^{\ell}$. Then all roots of $\mathscr{F}$ are isolated and the number of solutions to the system $\mathscr{F}$ is equal to the BKK bound.

Let f be a polynomial in ℓ variables, $\mathscr{A} = \mathrm{supp}(f)$, and $P = \mathrm{conv}(\mathscr{A}) \subseteq \mathbb{R}^{\ell}$.

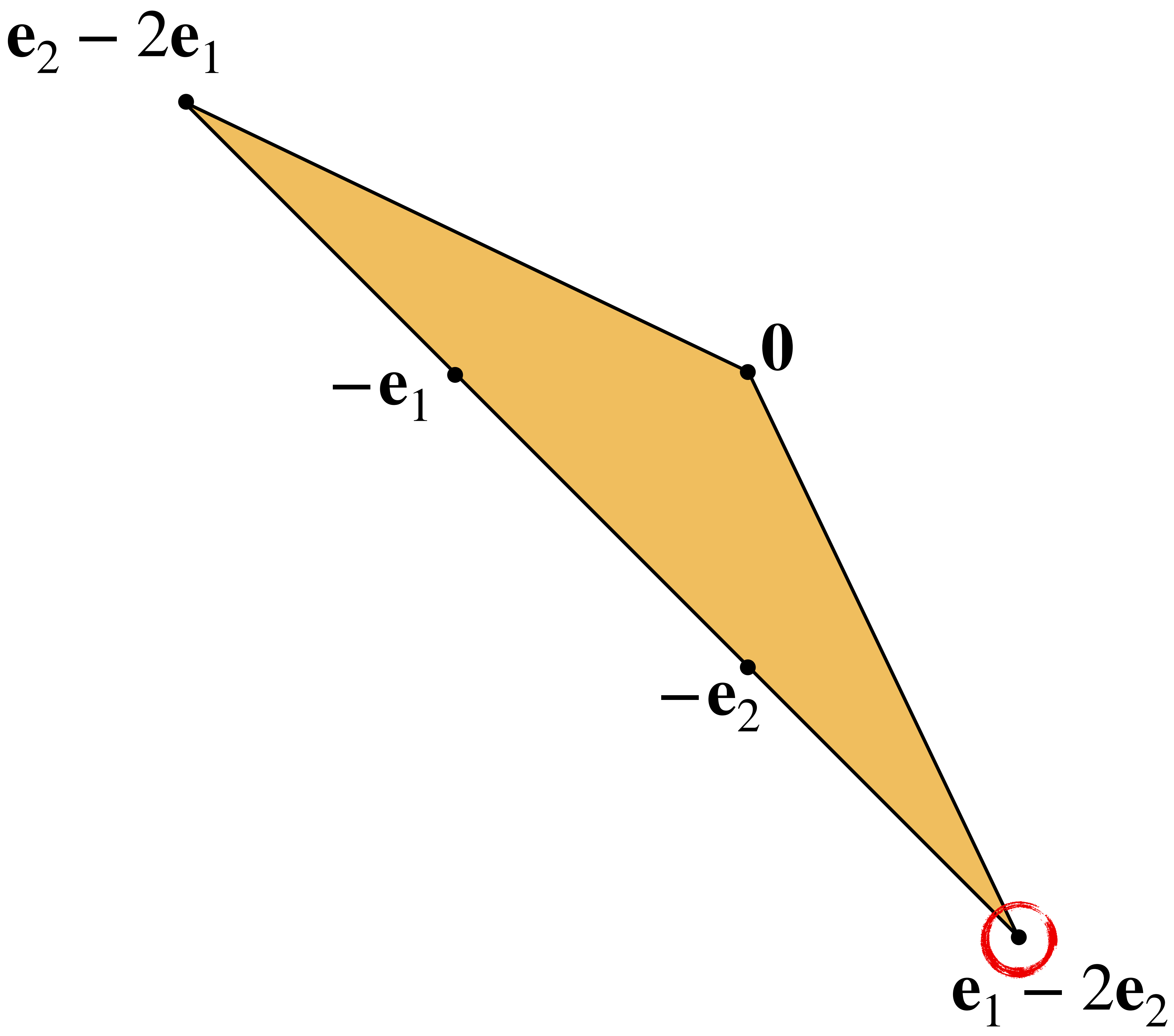
$$f(x, y) = 1 + x + y + xy \qquad P =$$



If $f(x) = \sum_{a \in \mathscr{A}} c_a x^a$, then the restriction of f to the face $F \subseteq P$ is $f(x) = \sum_{a \in \mathscr{A} \cap F} c_a x^a$.



BKK Discriminant: $\ell = 2$

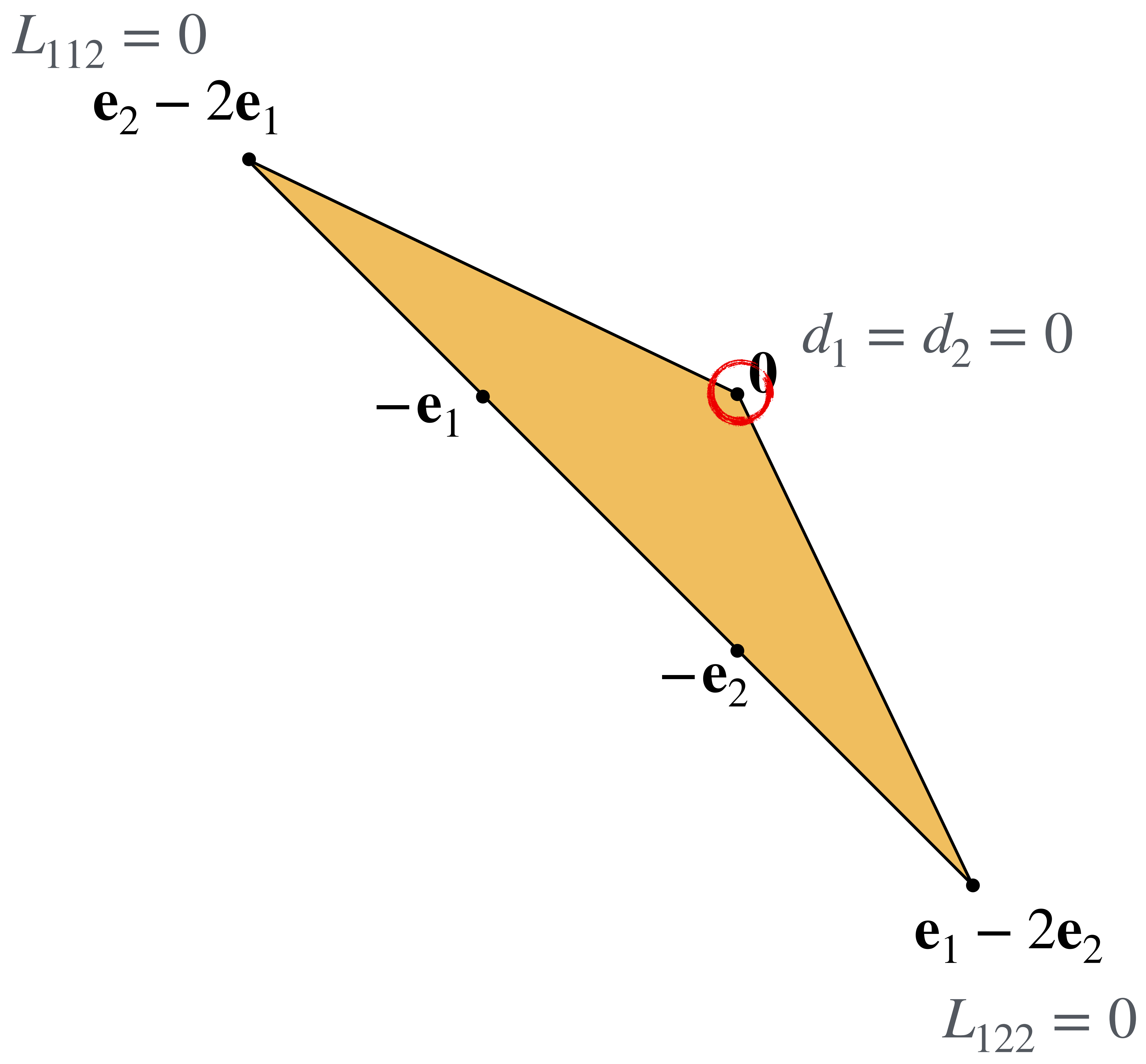


$$f_1(x) = L_{111}x_1^{-1} + 2L_{112}x_2x_1^{-2} - \boxed{L_{122}x_1x_2^{-2}} + 4d_1 = 0$$

$$f_2(x) = L_{222}x_2^{-1} + \boxed{2L_{122}x_1x_2^{-2}} - L_{112}x_2x_1^{-2} + 4d_2 = 0$$

$$-\frac{L_{122}x_1}{x_2^2} = 0$$
$$\frac{2L_{122}x_1}{x_2^2} = 0$$
$$\Longrightarrow \quad L_{122} = 0$$

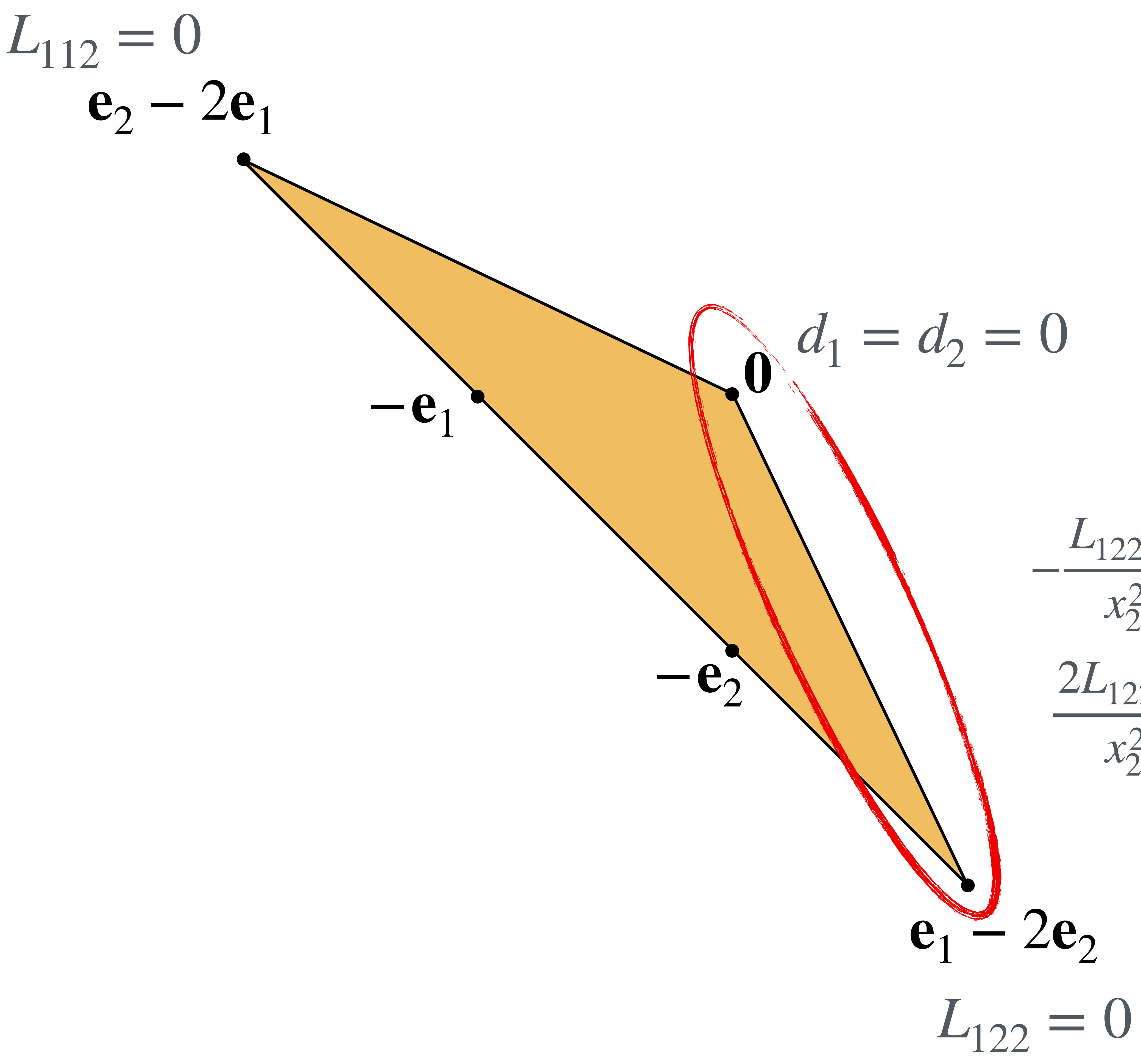
BKK Discriminant: $\ell = 2$



$$f_1(x) = L_{111}x_1{}^{-1} + 2L_{112}x_2x_1^{-2} - L_{122}x_1x_2^{-2} + \boxed{4d_1} = 0$$

$$f_2(x) = L_{222}x_2{}^{-1} + 2L_{122}x_1x_2^{-2} - L_{112}x_2x_1^{-2} + \boxed{4d_2} = 0$$

BKK Discriminant: $\ell = 2$

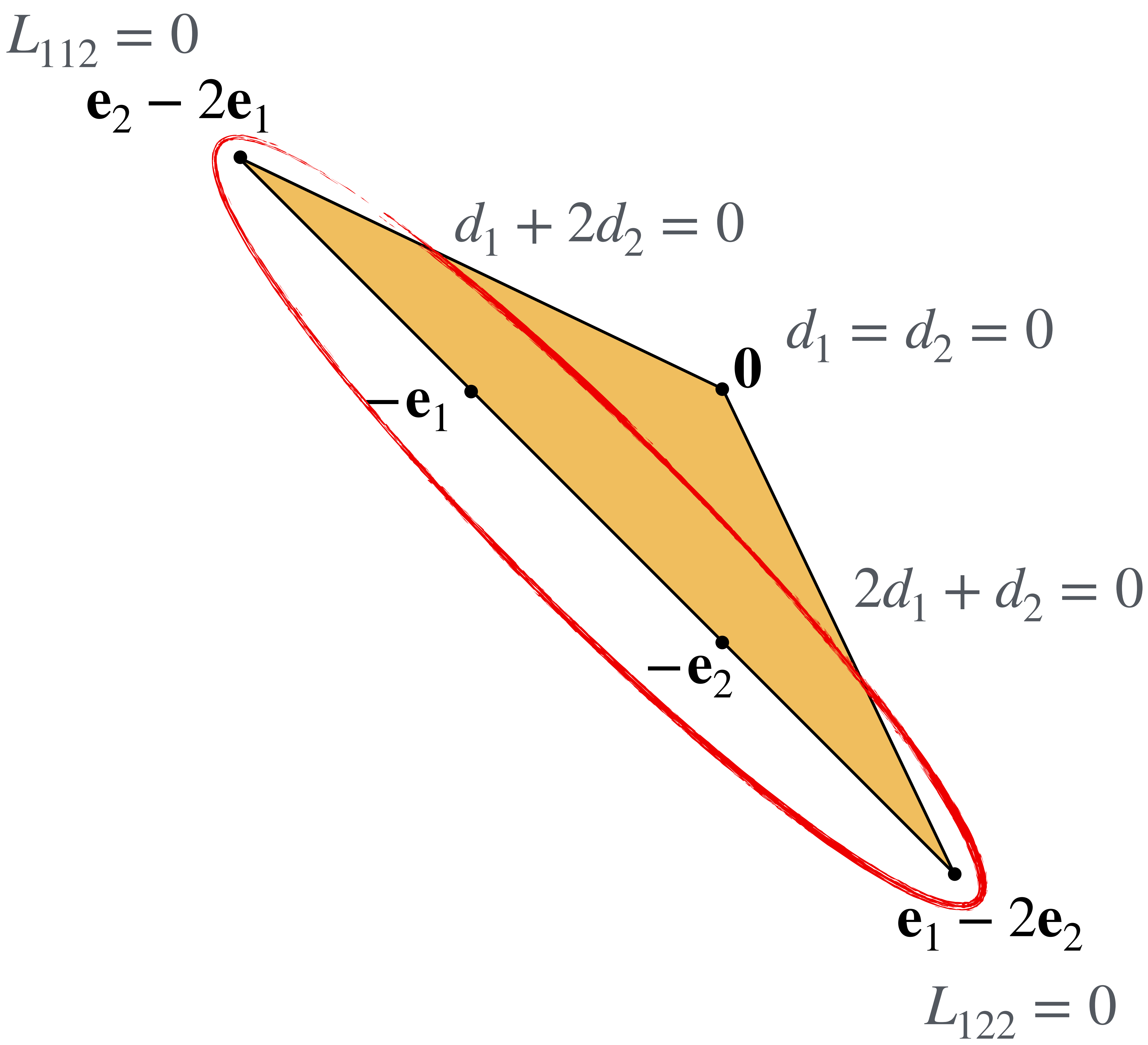


$$f_1(x) = L_{111}x_1^{-1} + 2L_{112}x_2x_1^{-2} - \boxed{L_{122}x_1x_2^{-2} + 4d_1} = 0$$

$$f_2(x) = L_{222}x_2^{-1} + \boxed{2L_{122}x_1x_2^{-2}} - L_{112}x_2x_1^{-2} + \boxed{4d_2} = 0$$

$$\begin{aligned} -\frac{L_{122}x_1}{x_2^2} + 4d_1 &= 0 \\ \frac{2L_{122}x_1}{x_2^2} + 4d_2 &= 0 \end{aligned} \implies 2d_1 + d_2 = 0$$

BKK Discriminant: $\ell = 2$

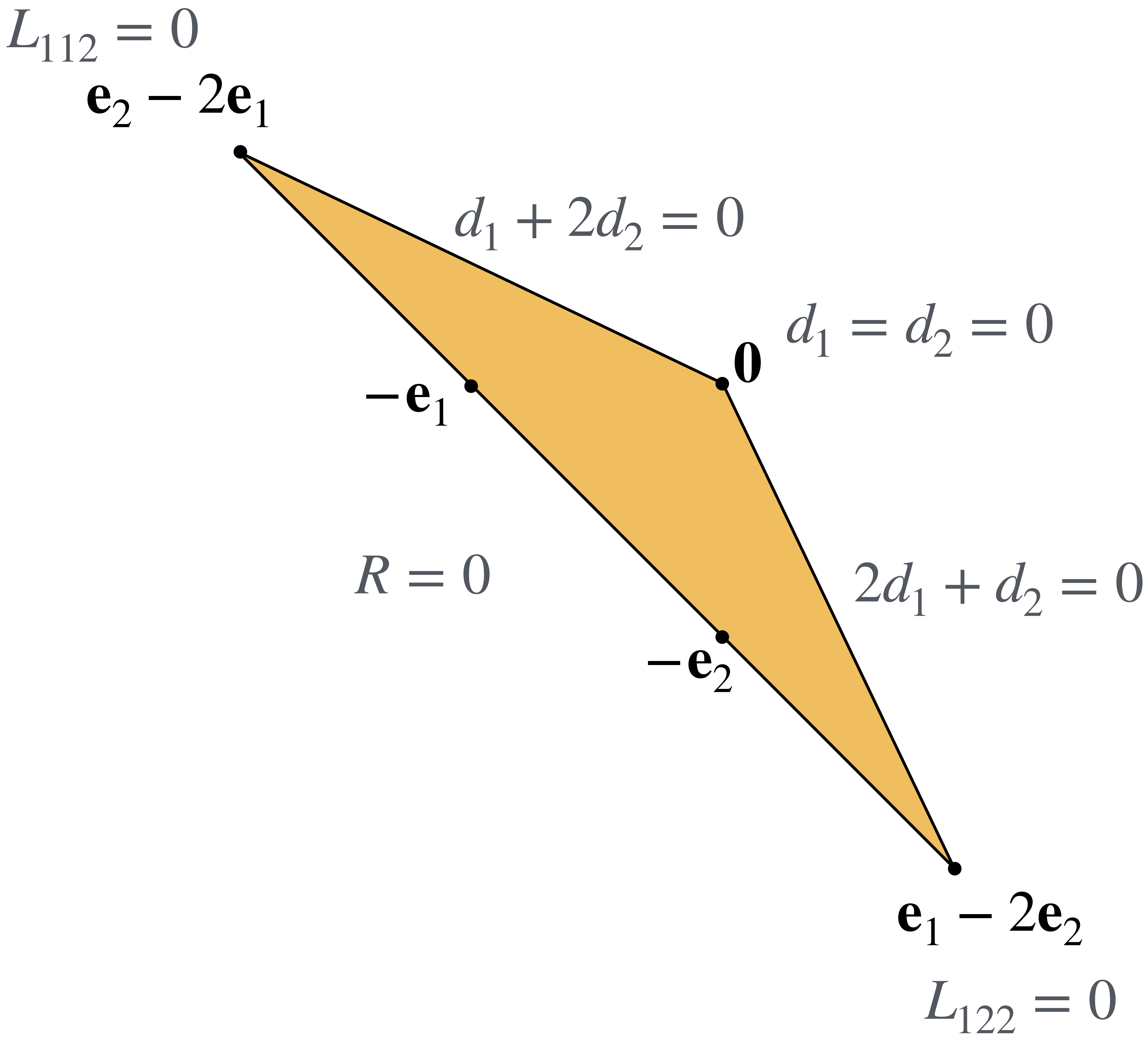


$$f_1(x) = \boxed{L_{111}x_1^{-1} + 2L_{112}x_2x_1^{-2} - L_{122}x_1x_2^{-2}} + 4d_1 = 0$$

$$f_2(x) = \boxed{L_{222}x_2^{-1} + 2L_{122}x_1x_2^{-2} - L_{112}x_2x_1^{-2}} + 4d_2 = 0$$

$$R = \begin{vmatrix} L_{122} & L_{222} & L_{111} & L_{112} & & \\ & L_{122} & L_{222} & L_{111} & L_{112} & \\ & & L_{122} & L_{222} & L_{111} & L_{112} \\ 3L_{122} & 2L_{222} & L_{111} & & & \\ & 3L_{122} & 2L_{222} & L_{111} & & \\ & & 3L_{122} & 2L_{222} & L_{111} & \end{vmatrix} = 0$$

BKK Discriminant: $\ell = 2$



Proposition (Bettiol-F, 2025). The system

$$f_1(x) = L_{111}x_1^{-1} + 2L_{112}x_2x_1^{-2} - L_{122}x_1x_2^{-2} + 4d_1 = 0$$
$$f_2(x) = L_{222}x_2^{-1} + 2L_{122}x_1x_2^{-2} - L_{112}x_2x_1^{-2} + 4d_2 = 0$$

has three solutions in $(\mathbb{C}^\times)^2$if the polynomial

$$(d_1 + 2d_2) \cdot (2d_1 + d_2) \cdot L_{112} \cdot L_{122} \cdot R$$

does not vanish.

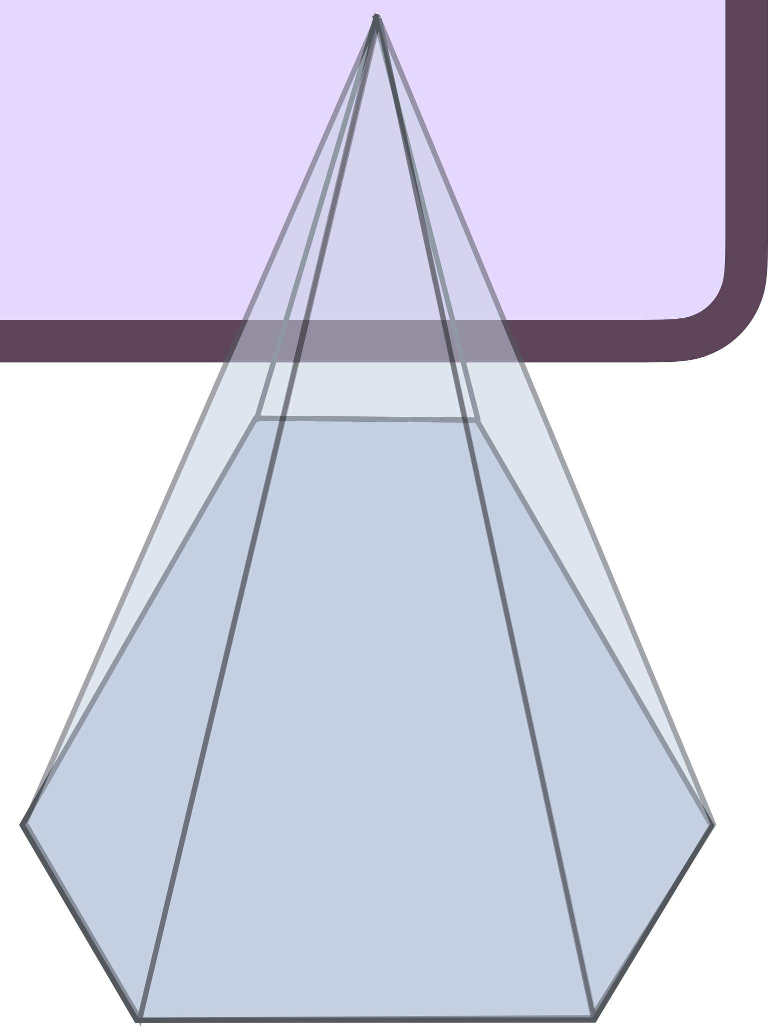
BKK Discriminant

Proposition (Bettiol-F, 2025). The system

$$f_i(x) = \frac{L_{iii}}{x_i} + \sum_{k \in [\ell] \setminus \{i\}} \left(\frac{2L_{iik}x_k}{x_i^2} - \frac{L_{ikk}x_i}{x_k^2} \right) + \sum_{j \neq k \in [\ell] \setminus \{i\}} 2L_{ijk} \left(\frac{x_j}{x_i x_k} + \frac{x_k}{x_i x_j} - \frac{x_i}{x_j x_k} \right) + 4d_i = 0, \quad i \in [\ell]$$

has $D_{\ell-1}$ many solutions in $(\mathbb{C}^\times)^\ell$ if the polynomial below does not vanish:

$$E_A(\mathrm{scal}) \cdot \prod_{\substack{S, T \subseteq [\ell] \\ S \cap T = \varnothing}} \left(\sum_{i \in S} 2d_i + \sum_{i \in T} d_i \right).$$



Scalar Curvature

The *scalar curvature* is the trace of the Ricci tensor.

$$f_i(x) = \overset{-4d_ir_i(x)}{\boxed{\frac{L_{iii}}{x_i} + \sum_{k\in[\ell]\backslash\{i\}} \left(\frac{2L_{iik}x_k}{x_i^2} - \frac{L_{ikk}x_i}{x_k^2} \right) + \sum_{j\neq k\in[\ell]\backslash\{i\}} 2L_{ijk} \left(\frac{x_j}{x_i x_k} + \frac{x_k}{x_i x_j} - \frac{x_i}{x_j x_k} \right)}} + 4d_i = 0, \quad i \in [\ell]$$

$$\mathrm{scal}(x) = \sum_{i=1}^{\ell} d_i r_i(x) = -\frac{1}{4} \left(\sum_{i=1}^{\ell} \frac{L_{iii}}{x_i} + \sum_{\substack{i,j,k=1 \\ k\neq i,j}}^{\ell} L_{ijk} \frac{x_k}{x_i x_j} \right)$$

$$x_i \frac{\partial}{\partial x_i} \mathrm{scal}(x) = -d_i r_i(x)$$

Principal A-Determinant

Let $\mathscr{A} \subseteq \mathbb{Z}^\ell$ be such that $f(x) = \sum_{a \in \mathscr{A}} c_a x^a$ is homogeneous. The principal A-determinant of f is a polynomial in the c_a defined as the resultant of the toric derivatives of f:

$$E_A(f) = R_A\left(x_1 \frac{\partial f}{\partial x_1}, \cdots, x_\ell \frac{\partial f}{\partial x_\ell}\right).$$

Familiar Resultant

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$R(f, f') = b^2 - 4ac$

Principal $\boldsymbol{A}$-Determinant of Scalar Curvature

Example ($\ell = 2$).

$$\mathrm{scal}(x) = -\frac{1}{4}\left(\frac{L_{111}}{x_1} + \frac{L_{222}}{x_2} + \frac{L_{122}x_1}{x_2^2} + \frac{L_{112}x_2}{x_1^2}\right)$$

$$E_A(\mathrm{scal}) = R_A\left(\frac{L_{111}}{x_1} + \frac{2L_{112}x_2}{x_1^2} - \frac{L_{122}x_1}{x_2^2},\ \frac{L_{222}}{x_2} + \frac{2L_{122}x_1}{x_2^2} - \frac{L_{112}x_2}{x_1^2}\right) = \begin{vmatrix} L_{122} & L_{222} & L_{111} & L_{112} & & \\ & L_{122} & L_{222} & L_{111} & L_{112} & \\ & & L_{122} & L_{222} & L_{111} & L_{112} \\ 3L_{122} & 2L_{222} & L_{111} & & & \\ & 3L_{122} & 2L_{222} & L_{111} & & \\ & & 3L_{122} & 2L_{222} & L_{111} & \end{vmatrix}$$

Principal A-Determinant

Let $\mathscr{A} \subseteq \mathbb{Z}^\ell$ be such that $f(x) = \sum_{a \in \mathscr{A}} c_a x^a$ is homogeneous.

Theorem (Gelfand-Kapronov-Zelevinsky, 1994). The principal A-determinant factors as

$$E_A(f) = \prod_{F \subseteq P} \Delta_{F \cap \mathscr{A}}^{m_F}(f)$$

where the product is over faces F of the polytope $P = \mathrm{conv}(\mathscr{A})$, the m_F are natural numbers and $\Delta_{F \cap \mathscr{A}}(f)$ is the defining polynomial of the A-discriminant:

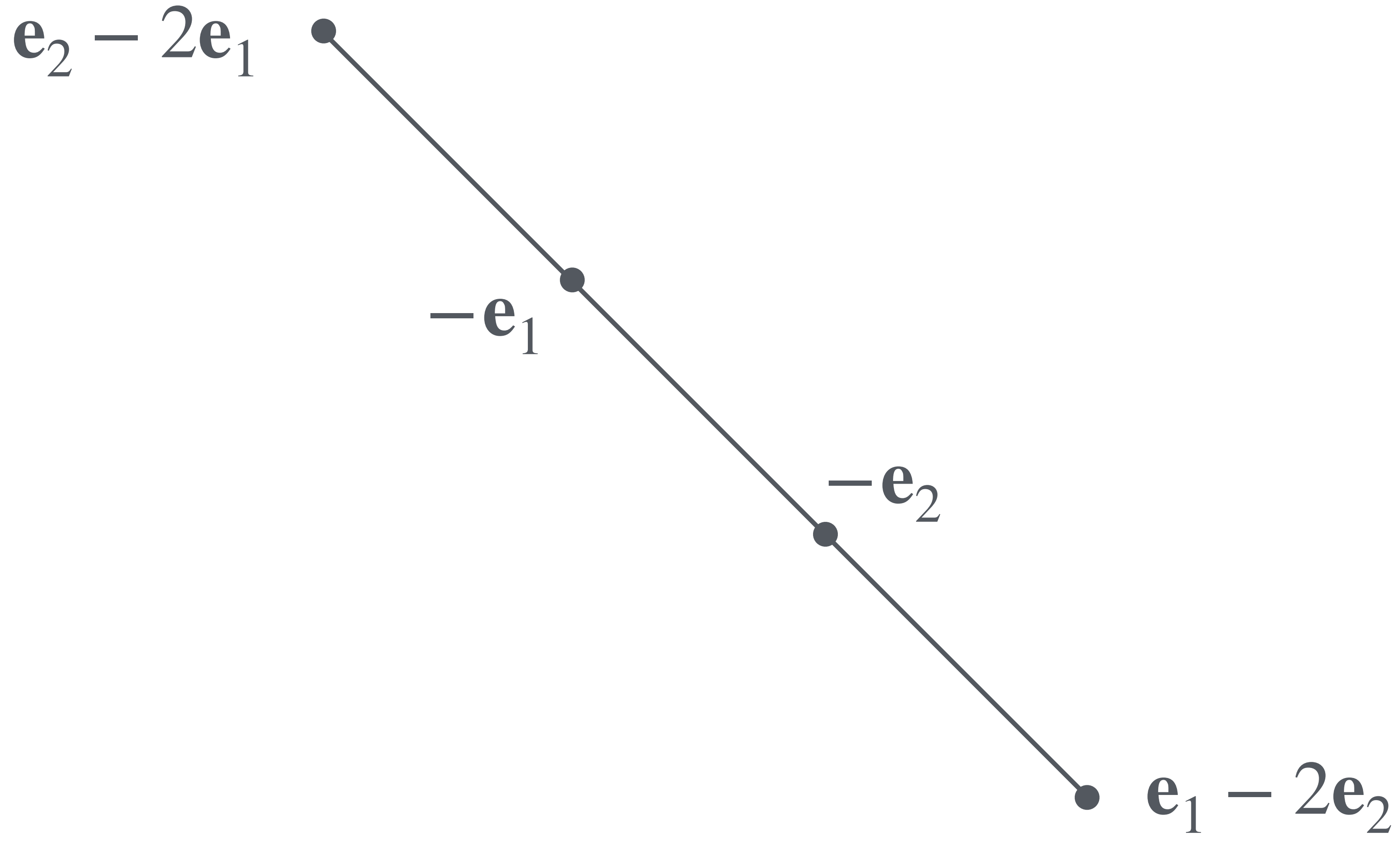
$$V(\Delta_{F \cap \mathscr{A}}(f)) = \overline{\left\{ c \in \mathbb{C}^{|\mathscr{A}|} : \text{there exists } x \in (\mathbb{C}^\times)^\ell \text{ such that } \frac{\partial f_F}{\partial x_i}(x) = 0 \text{ for } i \in [\ell] \right\}}$$

where f_F denotes the restriction of f to the face F.

Principal *A*-Determinant of Scalar Curvature

Example ($\ell = 2$).

$$\mathrm{scal}(x) = -\frac{1}{4}\left(\frac{L_{111}}{x_1} + \frac{L_{222}}{x_2} + \frac{L_{122}x_1}{x_2^2} + \frac{L_{112}x_2}{x_1^2}\right)$$



$$E_A(\mathrm{scal}) = \begin{vmatrix} L_{122} & L_{222} & L_{111} & L_{112} & & \\ & L_{122} & L_{222} & L_{111} & L_{112} & \\ & & L_{122} & L_{222} & L_{111} & L_{112} \\ 3L_{122} & 2L_{222} & L_{111} & & & \\ & 3L_{122} & 2L_{222} & L_{111} & & \\ & & 3L_{122} & 2L_{222} & L_{111} & \end{vmatrix}$$

$$= L_{112}L_{122}(L_{111}^2L_{222}^2 - 27L_{112}^2L_{122}^2 - 4L_{222}^3L_{112} - 4L_{111}^3L_{122} + 18L_{111}L_{222}L_{112}L_{122})$$

Geometric Corollary

Corollary (Bettiol-F, 2025) Let G/H be a compact homogeneous space whose isotropy representation consists of ℓ pairwise inequivalent irreducible summands. If the principal A-determinant $E_A(\mathrm{scal})$ does not vanish, then there are at most $D_{\ell-1}$ many G-invariant Einstein metrics on G/H. In particular, the Finiteness Conjecture holds on G/H.

Question (Bettiol-F, 2025) Construct geometric examples with $\ell > 2$ which achieve the BKK bound $D_{\ell-1}$.

Example: Full Flag Manifolds

Let $M = \mathsf{G}/\mathsf{H}$, where G is a compact simple Lie group of classical type and H is a maximal torus in G.

G	SU(3)	SU(4)	SU(5)	SU(6)	SO(5)	SO(7)	Sp(3)	SO(8)
BKK Bound	4	80	9,168	6,603,008	12	5,376	5,232	239,744
# complex solutions	4	59	7,908	5,037,448	10	4,224	4,512	150,256
# real solutions	4	29	1,596	191,252	6	750	728	11,128
# positive solutions, i.e., # G-invariant Einstein metrics	4	29	396	6,572	6	48	64	184
# isometry classes of G-invariant Einstein metrics	2	4	12	35	2	5	4	5

Thank you!