Grassmann and Flag Varieties in Linear Algebra, Optimization, and Statisics

An Algebraic Perspective

Hannah Friedman

UC Berkeley

SIAM AG 2025

based on joint work with Serkan Hoşten: arXiv 2505.15969

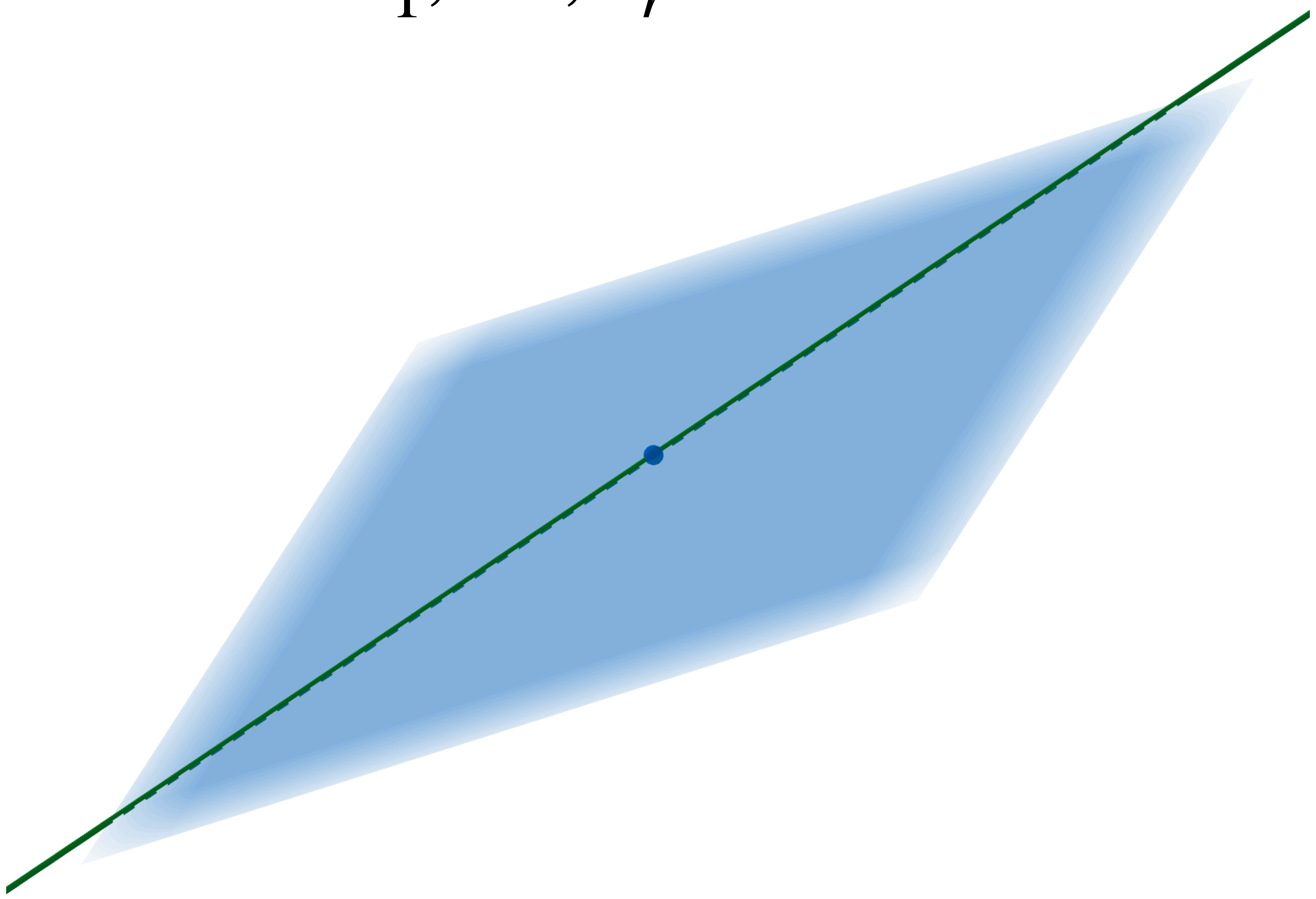
Flag Varieties

Definition. The flag manifold or variety

$$\mathrm{Fl}(k_1, \ldots, k_r; n) = \{W_1 \subseteq W_2 \subseteq \cdots \subseteq W_r \subseteq \mathbb{R}^n : \dim(W_i) = k_i, i = 1,\ldots,r\}$$

is the space of nested subspaces of dimension $k_1, \ldots, k_r$ in $\mathbb{R}^n$.

Example. A point in $\mathrm{Fl}(1,2;3)$.



How can we represent flags with polynomial equations?

The Many Lives of Flag Varieties: Stiefel Coordinates

Definition. The Stiefel manifold $V_{k,n}$ is the set of orthonormal k-frames

$$V_{k,n} = \{Z \in \mathbb{R}^{n \times k} : Z^T Z = \mathrm{Id}_k\} .$$

The orthogonal group $\mathrm{O}(n)$ is $V_{n,n}$.

Theorem (F.-Hoşten, 2025).

$$\dim(V_{k,n}) = \binom{n}{2} - \binom{n-k}{2} \qquad\qquad I(V_{k,n}) = \langle Z^T Z - \mathrm{Id}_k \rangle$$

The ideal $I(V_{k,n})$ is a complete intersection. When $k < n$, $I(V_{k,n})$ is prime.

See also: *The Degree of Stiefel Manifolds* by Brysiewicz and Gesmundo.

The Many Lives of Flag Varieties: Stiefel Coordinates

Definition. The Stiefel manifold $V_{k,n}$ is the set of orthonormal k-frames

$$V_{k,n} = \{Z \in \mathbb{R}^{n \times k} : Z^T Z = \mathrm{Id}_k\}\,.$$

The orthogonal group $\mathrm{O}(n)$ is $V_{n,n}$.

- $\mathrm{Gr}(k,n) = V_{k,n}/\mathrm{O}(k)$
- $\mathrm{Fl}(k_1,\ldots,k_r;n) = V_{k,n}/\mathrm{O}(k_1) \times \mathrm{O}(k_2-k_1) \times \cdots \times \mathrm{O}(k_r-k_{r-1})$

Example. $\mathrm{Fl}(1,2;3) \cong \mathrm{V}_{2,3}/\mathrm{O}(1)^2$

$$Z = (Z_1\ \ Z_2) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{pmatrix}$$

The Many Lives of Flag Varieties: Projection Coordinates

A linear subspace $W \subseteq \mathbb{R}^n$ is uniquely determined by the orthogonal projection onto W.

$$\mathrm{pGr}(k,n) = \{P \in \mathrm{Sym}(\mathbb{R}^n) \colon P^2 = P, \mathrm{trace}(P) = k\}$$

Example. $\mathrm{Fl}(1,2;3)$ $\qquad Z = (Z_1 \;\; Z_2) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{pmatrix}$

$$P_1 = Z_1 Z_1^T$$

projects onto W_1

$$P_2 = ZZ^T$$

projects onto W_2

$$P_2 P_1 = P_1$$

The Many Lives of Flag Varieties: Projection Coordinates

Theorem (F.-Hoşten, 2025). The projection flag variety $\mathrm{pFl}(k_1, \ldots, k_r; n)$ is smooth and has prime ideal

$$\langle P_i P_{i-1} - P_{i-1} : 2 \leq i \leq r \rangle + \langle P_i^2 - P_i, \ \mathrm{trace}(P_i) - k_i : 1 \leq i \leq r \rangle$$

Here $P_1, \ldots, P_r$ are symmetric $n \times n$ matrices. The ambient ring has $r\binom{n+1}{2}$ generators.

See also: "Optimization on Flag Manifolds" by Ye, Wong, and Lim

"The Two Lives of the Grassmannian" by Devriendt, F., Reinke, and Sturmfels

The Many Lives of Flag Varieties: Isospectral Coordinates

Goal: represent points in the flag with symmetric matrices.

Example. $\mathrm{Fl}(1,3;4) \cong V_{3,4}/\mathrm{O}(1)\times \mathrm{O}(2) \cong \mathrm{O}(4)/\mathrm{O}(1)\times \mathrm{O}(2)\times \mathrm{O}(1)$

$$Z = (Z_1 \;\; Z_2 \;\; Z_3) \mapsto \tilde{Z} = (Z_1 \;\; Z_2 \;\; Z_3 \;\; Z_4)$$

Fix $c_1 > c_2 = c_3 > c_4$.

$$\left\{ \tilde{Z} \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{pmatrix} \tilde{Z}^T : \tilde{Z} \in \mathrm{O}(4)/\mathrm{O}(1)\times \mathrm{O}(2)\times \mathrm{O}(1) \right\} = \left\{ Q \in \mathrm{Sym}(\mathbb{R}^n) : Q \text{ has spectrum } (c_1, c_2, c_3, c_4) \right\}$$

The fact that the set of isospectral matrices parameterizes flag varieties was first observed by Ye and Lim in *Simple Matrix Models for the Flag, Grassmann, and Stiefel Manifolds*.

The Many Lives of Flag Varieties: Isospectral Coordinates

Theorem (F.-Hoşten, 2025). Let $\mathrm{Fl}(k_1, \ldots, k_r; n)$ be a flag variety and let X be a symmetric matrix of unknowns. Given a generic choice of $c_1, \ldots, c_n$ satisfying $c_{k_j+1} = \cdots = c_{k_j+1}$ for $j = 0, \ldots, r$, the variety $\mathrm{Fl}_{\mathbf{c}}(k_1, \ldots, k_r; n)$ is smooth and its prime ideal is

$$\langle \prod_{j=1}^{r} (X - c_{k_j} \mathrm{Id}_n), \ \mathrm{trace}(X) - \sum_{j=1}^{n} c_j \rangle \, .$$

Example. If $r = 1$, $c_1 = \cdots = c_{k_1} = 1$, and $c_{k_1+1} = \cdots = c_n = 0$, then

$$\tilde{Z} \mathrm{diag}(1,\ldots,0) \tilde{Z}^T = ZZ^T = P \longrightarrow \mathrm{Fl}_{\mathbf{c}}(k_1; n) = \mathrm{pGr}(k_1, n) \, .$$

The Many Lives of Flag Varieties

$$
\begin{tikzcd}
 & \text{Plücker} \arrow[ddl, "P_i = \frac{k_i}{\mathrm{trace}(X_i^2)} X_i^2"'] & \\
 & \text{Stiefel} \arrow[u, "x_{i_1,\ldots,i_{k_s}} = \det(Z_{i_1\cdots i_{k_s},1\cdots k_s})"'] \arrow[dl, leftrightarrow, "\boxed{P_i = Z_{1\cdots n,1\cdots k_i} Z^T_{1\cdots n,1\cdots k_i}}"] \arrow[dr, leftrightarrow, "\boxed{S = \tilde{Z}\,\mathrm{diag}(c_1,\ldots,c_n)\,\tilde{Z}^T}"] & \\
\text{Projection} \arrow[rr, "S = c_n\mathrm{Id}_n + (c_{k_r} - c_n)P_{k_r} + \cdots + (c_{k_1} - c_{k_2})P_1"'] & & \text{Isospectral}
\end{tikzcd}
$$

orthogonal $n \times k$ matrix

list of $n \times n$ projection matrices

symmetric $n \times n$ matrix

Figure 1: Diagram explaining how to move from one life of the flag variety to another. If $A \to B$ in the diagram, the edge label explains how to write the B coordinates in terms of the A coordinates. Two of the arrows are bidirectional, meaning that one direction comes from matrix multiplication and the other comes from a matrix factorization.

Algebraic Degree of an Optimization Problem

Optimization Problem

optimize $f(\mathbf{x})$

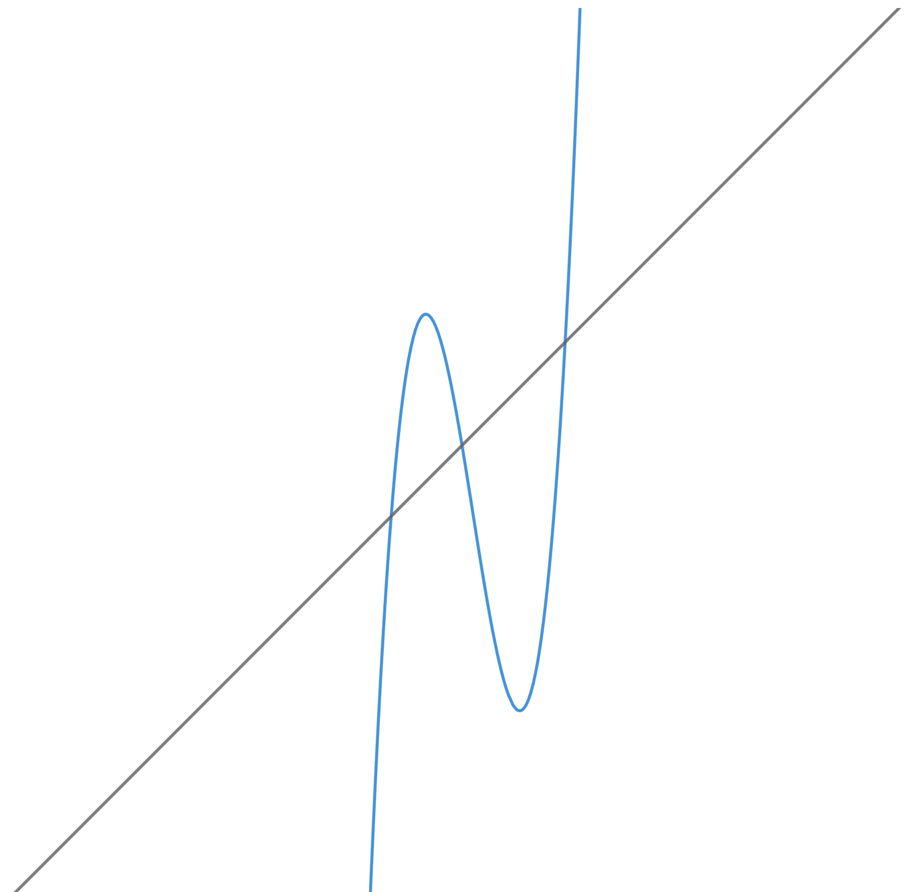
subject to $G(\mathbf{x}) = 0$

Critical Points

$$\operatorname{rank}\big(\operatorname{Jac}(G(\mathbf{x})) \mid \nabla f(\mathbf{x})\big) = \operatorname{rank} \operatorname{Jac}(G(\mathbf{x}))$$
$$G(\mathbf{x}) = 0$$

often 0-dimensional

Definition. The *algebraic degree* of an optimization problem is the number of critical points.



When the variety is not zero dimensional, its degree can still give an idea of the complexity of the problem.

The algebraic degree of a problem is a proxy for the difficulty of correctly solving the problem.

Multi-Eigenvalue Problem

Let A be real, symmetric $n \times n$ matrix.

Goal: compute an $n \times k$ matrix $Z \in V_{k,n}$ such that the columns of Z are eigenvectors of A.

$$\max_{Z \in V_{k,n}} \operatorname{trace}(Z^T A Z)$$

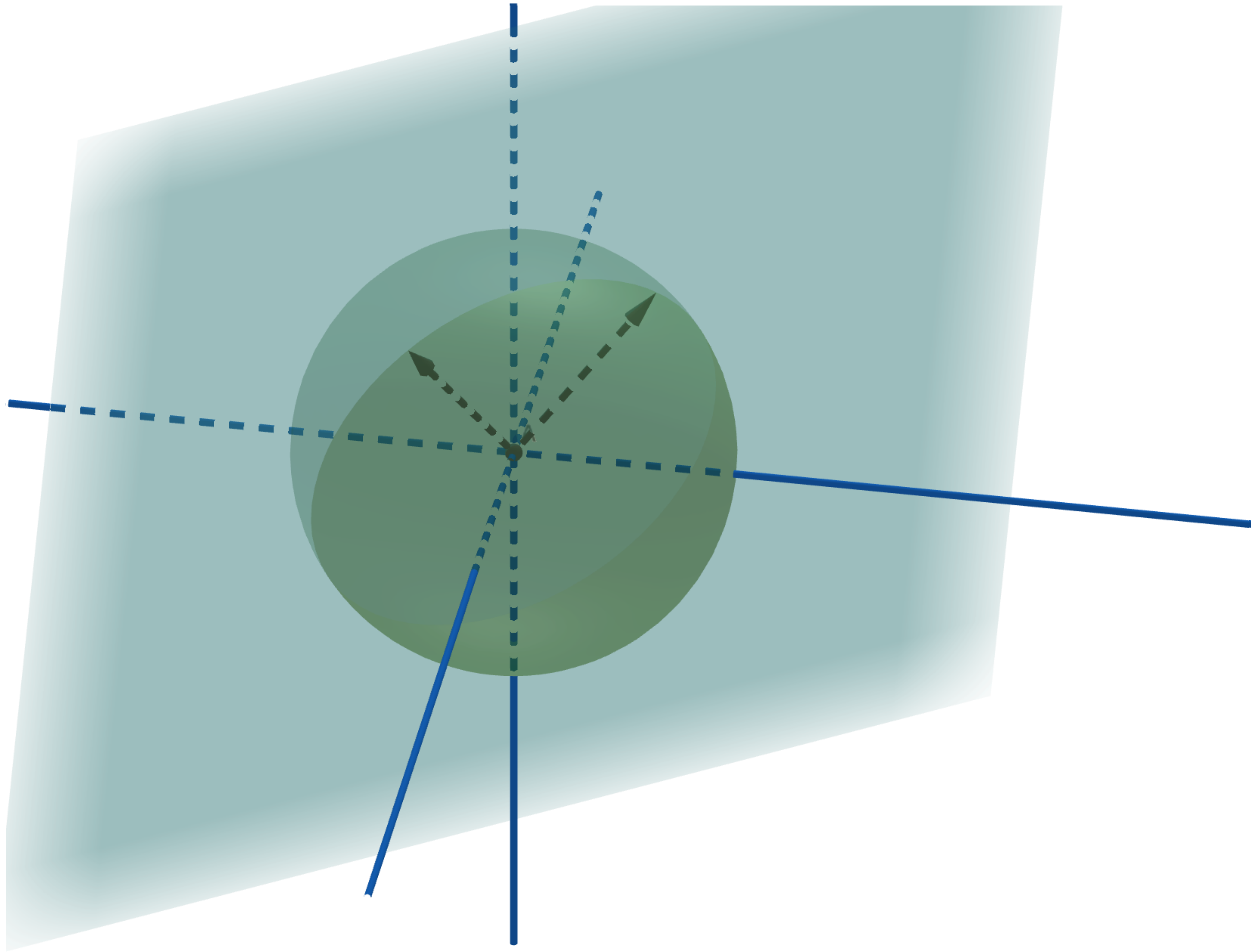
Critical Points of the Multi-Eigenvalue Problem

The optimization problem $\max_{Z \in V_{k,n}} \mathrm{trace}(Z^T A Z)$ is invariant under the action of $\mathrm{O}(k)$.

Let $Q \in \mathrm{O}(k)$.

$$\mathrm{trace}(Q^T Z^T A Z Q) = \mathrm{trace}(Z^T A Z Q Q^T) = \mathrm{trace}(Z^T A Z)$$

$$Z^T Z = \mathrm{Id}_k \implies Q^T Z^T Z Q = Q^T Q = \mathrm{Id}_k$$



Critical Points of the Multi-Eigenvalue Problem

Theorem (F., Hoşten 2025). Let A be a generic real symmetric $n \times n$ matrix and let $Z \in V_{k,n}$. The algebraic set of complex critical points of the eigenvalue optimization problem is

$$\bigsqcup_{\{i_1,\ldots,i_k\}\in\binom{[n]}{k}} \{[u_{i_1}\, u_{i_2}\, \cdots\, u_{i_k}]\, Q \,:\, Q \in \mathrm{O}(k)\}$$

where $q_1, \ldots, q_n$ is an orthonormal eigenbasis of A. This algebraic set is a disjoint union of $\binom{n}{k}$ irreducible varieties isomorphic to $\mathrm{O}(k)$, and hence its dimension is equal to $\dim(\mathrm{O}(k))$ and its degree is equal to $\deg(\mathrm{O}(k)) \cdot \binom{n}{k}$.

Multi-Eigenvalue Problem in Projection Coordinates

$$\max_{Z \in V_{k,n}} \operatorname{trace}(Z^T A Z) \qquad \max_{Z \in V_{k,n}} \operatorname{trace}(A Z Z^T) \xrightarrow[ZZ^T \mapsto P]{} \max_{P \in \mathrm{pGr}(k,n)} \operatorname{trace}(AP)$$

$\{i_1, \ldots, i_k\}$ $\{i_1, \ldots, i_k\}$

Multi-Eigenvalue Problem in Projection Coordinates

Theorem (F.-Hoşten, 2025). Let A be a generic real symmetric $n \times n$ matrix. The optimization problem

$$\max_{P \in \mathrm{pGr}(k,n)} \quad \mathrm{trace}(AP)$$

has critical point set

$$\left\{ [u_{i_1}\, u_{i_2} \; \cdots \; u_{i_k}][u_{i_1}\, u_{i_2} \; \cdots \; u_{i_k}]^T \mid \{i_1, \ldots, i_k\} \in \binom{[n]}{k} \right\}$$

where $u_1, \ldots, u_n$ is an orthonormal eigenbasis of A and algebraic degree $\binom{n}{k}$.

Corollary. The linear optimization degree of $\mathrm{pGr}(k, n)$ is $\binom{n}{k}$.

The linear optimization degree of a variety was introduced in *Linear Optimization on Varieties and Chern Mather Classes* by Maxim, Rodriguez, Wang, and Wu.

Multi-Eigenvalue Problem in Isospectral Coordinates

$$P = ZZ^T = \tilde{Z}\ \mathrm{diag}(1,\ldots,1,0,\ldots,0)\ \tilde{Z}^T \qquad X = \tilde{Z}\ \mathrm{diag}(c_1, c_2, \ldots, c_n)\ \tilde{Z}^T$$

$$\max_{P \in \mathrm{pGr}(k,n)} \quad \mathrm{trace}(AP) \qquad \bigstar \max_{X \in \mathrm{Fl}_{\mathbf{c}}(\mathbf{k};n)} \quad \mathrm{trace}(AX)$$

Theorem (F.-Hoşten, 2025). The critical points of $\bigstar$ are the points in $\mathrm{Fl}_{\mathbf{c}}(\mathbf{k}; n)$ representing different flag structures on the eigenspaces of A. The degree of $\bigstar$ is

$$\binom{n}{k_1, k_2 - k_1, \ldots, n - k_r}.$$

Corollary. The linear optimization degree of $\mathrm{Fl}_{\mathbf{c}}(\mathbf{k}; n)$ is $\binom{n}{k_1, k_2 - k_1, \ldots, n - k_r}$.

Heterogeneous Quadratics Minimization Problem

Problem. Fix real symmetric matrices $A_1, \ldots, A_k$. How many critical points does the following optimization problem have?

$$\min_{Z \in V_{k,n}} \sum_{i=1}^{k} Z_i^T A_i Z_i \qquad Z = (Z_1 \;\; Z_2 \;\; \cdots \;\; Z_k)$$

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$k=2$	8	40	112	240	440	728	1120	1632
$k=3$		80	960	5536	21440	64624		
$k=4$			1920	57216				

Table 1: Degrees of the heterogeneous quadratics minimization problem for small k, n.

Conjecture. The number of critical points for $k = 2$ is $8 \sum_{j=1}^{n-1} j^2$.

Stiefel & Projection Coordinates

$$V_{k,n} \to \mathrm{pFl}(1,2,\ldots,k;n)$$

$$Z = (Z_1\ \ Z_2\ \ \cdots\ \ Z_k) \mapsto (Z_1Z_1^T,\ Z_1Z_1^T + Z_2Z_2^T,\ \ \ldots\ ,\ ZZ^T) = (P_1, P_2, \ldots, P_k)$$

$$\min_{Z\in V_{k,n}} \sum_{i=1}^{k} Z_i^T A_i Z_i$$

$$\min_{Z\in V_{k,n}} \sum_{i=1}^{k} \mathrm{trace}(A_i Z_i Z_i^T)$$

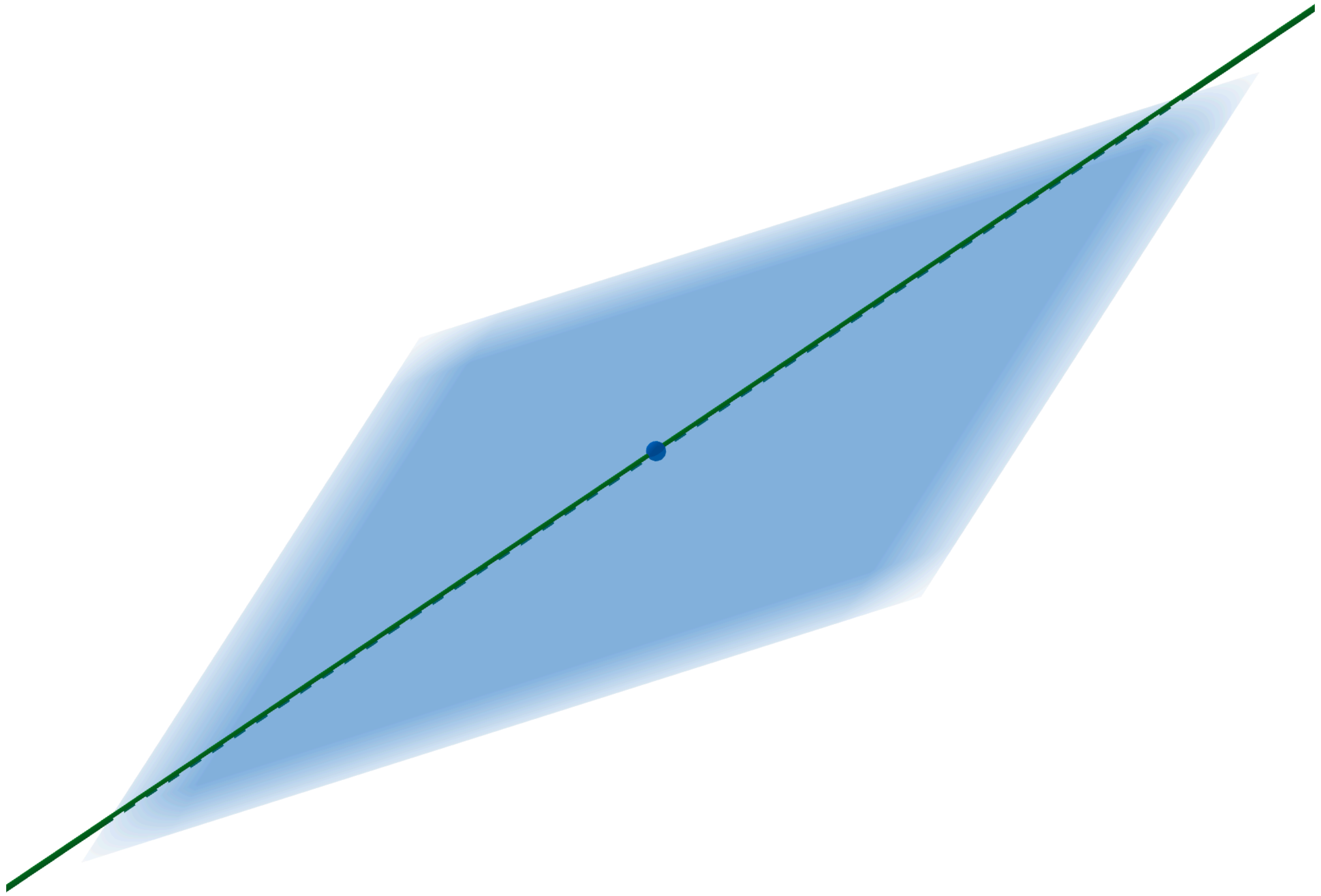
$$\longrightarrow$$

$$\min_{(P_1,\ldots,P_n)\in\mathrm{pFl}(\mathbf{k};n)} \sum_{i=1}^{k} \mathrm{trace}(B_iP_i)$$

$$\mathbf{k} = (1,2,\ldots,k)$$

$\frac{1}{2^k}$ Algebraic degree of $\min_{Z\in V_{k,n}} \sum_{i=1}^{k} Z_i^T A_i Z_i$ $=$ Algebraic degree of $\min_{(P_1,\ldots,P_n)\in\mathrm{pFl}(\mathbf{k};n)} \sum_{i=1}^{k} \mathrm{trace}(B_iP_i)$ $=$ Linear optimization degree of $\mathrm{pFl}(\mathbf{k};n)$

Thank you!



arXiv 2505.15969

The Many Lives of Flag Varieties: Plücker Coordinates

Theorem. The variety $\mathrm{Fl}(k_1, \ldots, k_r; n) \subseteq \mathbb{P}^{\binom{n}{k_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{k_r}-1}$ in Plücker coordinates is defined by the prime ideal generated by the quadrics

$$x_{i_1,\ldots,i_{k_s}} x_{j_1,\ldots,j_{k_t}} - \sum x_{i',\ldots,i'_{k_s}} x_{j'_1,\ldots,j'_{k_t}}$$

for every pair $1 \leq s < t \leq r$ and where the sum is over all (i', j') obtained by exchange the first m of the j-subscripts with m of the i-subscripts while preserving their order.